Would ambiguity averse investors hedge risk in equity markets?

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Abstract

This paper studies the conjecture that investors prefer derivative markets over the equity market when hedging risks. An investor who wants to hedge, say inflation or crash risk, faces substantially more beta uncertainty in the stock market than in the derivatives market. We show theoretically, that an investor with smooth ambiguity aversion preferences avoids a hedge portfolio consisting of stocks, which is typically subject to large beta uncertainty. The ambiguity averse investor prefers to hedge using derivatives (TIPS and options) which are not subject to beta uncertainty. More specifically, we show that equilibrium risk premiums for assets with large beta uncertainty (long-short portfolio of stocks) decline once derivatives with less beta uncertainty (TIPS and options) are introduced. In line with this theory, we find that the inflation risk premium in the equity market disappears after TIPS were introduced.

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1 Introduction

Over the past decades, a rapidly growing number of papers has discovered new risk factors that explain cross-sectional variation in stock returns - a phenomenon that was recently dubbed as the "factor zoo" (Harvey, Liu, and Zhu (2016) and McLean and Pontiff (2016)). The presence of some risk premiums (e.g. inflation and crash risk premiums) implies that investors' hedging demand is sufficiently large to move stock prices. This finding is striking, because inflation and crash risk are more directly hedged using TIPS and out-of-the-money put options than using long-short equity portfolios. Put differently, an investor faces substantial beta uncertainty and practical difficulties when hedging risk in the stock market. The investor should estimate factor exposures, create long-short portfolios and re-balance positions in a timely manner. Factor exposures are often poorly estimated and time-varying in an unpredictable way which leads to a different *ex post* exposure than *ex ante* desired. Moreover, many stocks are difficult to short and re-balancing is time-consuming and costly. These uncertainties and practical difficulties play less role in option or TIPS markets. We therefore ask the question whether the availability of more direct hedges such as stock options or TIPS affects stock market risk premiums.

We first study a model with two agents determining the risk premium of a long-short equity (hedge) portfolio in equilibrium. Both agents have smooth ambiguity aversion preferences. We think of the first agent as an institutional investor that is exposed to an exogenous shock and seeks a hedge. For example, to limit value-at-risk or meet regulatory constraints. We assume that the first agent observes each stock's expected payoffs and the beta of the long-short portfolio with respect to the exogenous shock. We think of the second agent as a hedge fund that has less exposure to the exogenous shock. The second agent does not observe expected stock returns and is willing to trade stocks thereby enabling the hedge, if the risk premium is large enough. In the second stage, we expand the investment universe with a derivative (TIPS or options). The derivative is characterized as an asset with significantly less beta uncertainty. We solve for equilibrium demands, asset prices and risk premiums in the initial and expanded universe.

First, we show theoretically that the long-short portfolio carries an uncertainty premium on top of the standard equity risk premium when agents cannot trade the direct hedge (derivative)

[1]. Ambiguity-averse investors who cannot trade derivatives thus require higher returns when beta uncertainty increases. Second, we document a re-allocation of wealth from the long-short equity portfolio to derivatives, once we allow derivative trading. This re-allocation can be contributed to (i) less hedging (beta) uncertainty and (ii) diversification benefits. We also show that the risk premium of the long-short portfolio decreases in absolute terms when we allow derivative trading [2]. This finding is consistent with the idea that investors who are seeking a hedge, prefer assets with less hedging uncertainty over ones with more hedging uncertainty. Using a representative set of parameters for the universe of stocks and derivatives, we show that the risk premium of the long-short portfolio reduces from approximately 8% to 6.75% (Figure 1) when introducing a derivative.

We also test our main predictions empirically. Following our line of thought, we should observe a decline in stock market risk premium when derivative assets are introduced in financial markets. We test this conjecture with the introduction of TIPS in January 1997. More specifically, we find that the inflation risk premium in equity markets shows a decrease approximately 0.5% per month when TIPS become available for investors. This change in premium cannot be explained by the standard Fama-French factors.

We assume that agents are ambiguity averse for two reasons. First, there is large empirical evidence that agents are averse to uncertainty regarding the parameters that govern the payoff distribution (e.g., the Ellsberg (1961) paradox). Second, ambiguity averse agents demand larger risk premiums than non-ambiguity averse agents when assets (such as stocks) are in positive net supply. Higher baseline risk premiums thus serve the main purpose of our paper: showing that stock market risk premiums decrease when agents can hedge risks more directly using options.

This paper contributes to the strand of literature that favors higher hurdles for accepting risk factors (e.g. Lewellen, Nagel, and Shanken (2008), Harvey, Liu, and Zhu (2016) and Linnainmaa and Roberts (2017)). We show that hedging-related risk premiums in the equity market decrease when investors can hedge the risk in an alternative market with less beta uncertainty. In other words, we cast doubt on the level of hedging-related risk premiums in the equity market. These

[1]See Theorem 3.4
[2]See Theorem 4.3

premiums seems strikingly large because investors have access to preferred hedging opportunities such as TIPS or options.

This paper also adds to the literature on beta uncertainty which documents both positive and negative relations between beta uncertainty and risk premiums. On the one hand, Gollier (2016) shows that uncertainty in a project's consumption CAPM beta *increases* expected returns. Likewise, Maenhout (2004) finds that a robust investor demands a *higher* equity risk premium when expected returns are uncertain. On the other hand, Barahona, Driessen, and Frehen (2019) document that an ambiguity averse agent's demand and therefore the risk premium *reduces* when beta (hedging) uncertainty increases. The main difference between Barahona, Driessen, and Frehen (2019) and the previous literature is the assumption that hedge assets are in zero net supply. As a result of that assumption, investors can reduce their demand for the hedge asset when beta uncertainty increases without the constraint that aggregate demand needs to be larger than zero. Our work differs because in Barahona, Driessen, and Frehen (2019) the ambiguous agent is also the one who seeks the hedge, while in our paper the ambiguous agent is the one who provides the hedge. That allows us to study how an investor behaves when having access to both *direct* and *indirect* hedge assets. This setup thus serves the main purpose of our paper: to test the conjecture that investors prefer direct hedges over indirect ones. Moreover, we model ambiguity aversion using Klibanoff, Marinacci, and Mukerji (2005)'s smooth ambiguity aversion preferences rather than Gilboa and Schmeidler (1989)'s max-min utility which relies heavily on extreme outcomes.

Our paper contributes to the ambiguity aversion literature by creating a setting with two representative agents and ambiguity about expected payoffs and betas between payoffs and the exogenous shock. More specifically, we rely on the Klibanoff, Marinacci, and Mukerji (2005) smooth ambiguity aversion framework and employ Maccheroni, Marinacci, and Ruffino (2013)'s approximation and solve for equilibrium demands, prices and risk premiums. Our paper thus extends Ruffino (2014)'s one-agent model by incorporating uncertainty about the beta of the asset with an underlying risk factor. In order to keep notation simple, albeit in line with the literature, we refer to uncertainty as a situation in which the agent does not know the parameters that govern the distribution of payoffs, while ambiguity is a part of the agent preference structure.

This paper is organized as follows. Section 2 presents the main assumptions underlying our theoretical framework. Section 3 is the analysis of the model *without* derivatives and Section 4 is the analysis *with* derivatives. Section 5 provides empirical evidence and Section 6 concludes.

2 Financial markets and preferences

In this section we present the general assumptions underlying our model. We first define agent preferences and then discuss the two markets that agents can trade in. In the first, agents can only trade a long-short portfolio - an asset *with* parameter uncertainty - to hedge risks. In the second, agents can trade both the long-short portfolio and a derivative *without* parameter uncertainty. Finally, we define the two types of agents in our markets. The types are different in terms of the information they possess and their exposure to an exogenous shock.

2.1 Preferences

We employ the smooth ambiguity preferences of Maccheroni, Marinacci, and Ruffino (2013). This framework decouples risk-aversion from ambiguity aversion in an analytically tractable way. The ambiguity in our case is expressed by parameter uncertainty. Let θ be a vector of parameters with a (joint) probability distribution with a density that we denote by f. In our model θ will be a vector of parameters of the mean of the payoff distribution and the beta of the payoff with the risk factor. The distribution of wealth depends on the realization of a vector θ.[3] We refer to $f(\theta)$ as the density of the uncertainty distribution. The value function of an ambiguity averse investor, as a function of wealth W, can then be expressed as

$$\begin{aligned} V(W) \;=\; & \int E(W|\theta) f(\theta)\, d\theta - \frac{\gamma}{2}\int Var(W|\theta) f(\theta)\, d\theta \\ & -\frac{\delta}{2}\int \left(E(W|\theta) - \int E(W|\theta) f(\theta)\, d\theta\right)^2 f(\theta)\, d\theta, \end{aligned} \tag{1}$$

where $\gamma > 0$ denotes the risk-aversion and $\delta \geq \gamma$ denotes the ambiguity-aversion parameter. This preference structure is similar to the mean-variance framework. The first term in (1) is the expectation of the expected wealth, given θ, calculated with respect to the uncertainty distribution of θ. The second term is the expectation of the variance of wealth, given θ, also calculated with respect to the uncertainty distribution. This term is multiplied by the risk-aversion $\gamma/2$. The

[3] Thus, the distribution of wealth can be seen as a mixture distrbution where the mixture is done by the distribution of parameters $f(\theta)$

last term is the variance of the expected wealth, given theta, again calculated with respect to the uncertainty distribution. This term is multiplied by the ambiguity-aversion parameter $\delta/2$. According to Maccheroni, Marinacci, and Ruffino (2013), an aversion to ambiguity arises when $\delta > \gamma$. In the case of $\delta = \gamma$, the agent is ambiguity neutral, and the preference structure collapses to regular mean-variance. In this case, the mean and variance are calculated with respect to the predictive (mixture) distribution of W. Intuitively, if different values of θ produce very different values of $E(W|\theta)$, the agent faces large parameter uncertainty and the last term will be relatively large. Equation (1) is the main workhorse of our model.

2.2 Market structure - hedging with stocks only

The market consists of a non-tradable risk with a cash-flow X_Q and three tradable assets: (i) a positive net supply market portfolio with a payoff X_M, (ii) a zero net supply long-short hedge portfolio with a payoff X_H and (iii) an infinite supply risk-free asset with a (gross) rate of return R_F. We first consider a market where agents can only hedge the non-tradable risk using the long-short portfolio of stocks with beta uncertainty. Denote by P_M and P_H the market prices of the market portfolio and the long-short portfolio respectively. For brevity, we will refer to this case as the "restricted" case. We define long-short hedge portfolios in the empirical exercise as a special case of the theoretical hedge portfolios. In the model, we do *not* assume that the long-short portfolio is self-financing.[4] As a result, the hedge portfolio can have a non-zero price. In the empirical part, we follow the literature and define the hedge portfolio as self-financing.

Denote $m_H \equiv E(X_H)$ and $b_H \equiv \frac{cov(X_Q, X_H)}{Var(X_H)}$. These are the two uncertain parameters in our model. Conditionally on these two parameters, the payoffs in the economy follow a multivariate Normal distribution.

$$\begin{pmatrix} X_M \\ X_H \\ X_Q \end{pmatrix} \sim_{|m_H, b_H} N \left[\begin{pmatrix} \mu_M \\ m_H \\ \mu_Q \end{pmatrix}, \begin{pmatrix} \sigma_M^2 & 0 & 0 \\ 0 & \sigma_H^2 & b_H \sigma_H^2 \\ 0 & b_H \sigma_H^2 & \sigma_Q^2 \end{pmatrix} \right], \tag{2}$$

[4]Note that we do assume that the long-short portfolio is in zero net supply.

with σ_M^2, σ_H^2 and σ_Q^2 the variance of the payoff of the market asset, the hedge asset, and the exogenous risk respectively. Moreover, $b_H\sigma_H^2$ is the covariance between the hedge long-short portfolio and the exogenous risk. The parameters m_H and b_H are uncertain, and are drawn from an independent multivariate Normal distribution,

$$\begin{pmatrix} m_H \\ b_H \end{pmatrix} \sim N\left[\begin{pmatrix} \eta_H \\ \beta_H \end{pmatrix}, \begin{pmatrix} \xi_{m_H}^2 & 0 \\ 0 & \xi_{b_H}^2 \end{pmatrix}\right], \tag{3}$$

where ξ_{m_H} and ξ_{b_H} parameterize the mean and beta uncertainty. An intuitive way to think about it is that nature draws two parameters from the distribution in (3), agents trade (which will be analyzed in Section 3), and finally all the payoffs follow a distribution that is defined in (2). For analytical simplicity, we assume that the market asset's cash-flow X_M is independent from all the other cash-flows in the model. Put differently, we assume that the market betas of the long- and short leg of the hedge portfolios are equal and the hedge portfolio is thus market neutral. We also assume (for simplicity) that there is no uncertainty about the payoffs of the market portfolio and the exogenous risk. Assuming uncertainty about the mean of these two payoffs will not affect the main conclusions of our paper. We assume that the mean and beta of the hedge asset are the only uncertain parameters. These two sources of uncertainty are sufficient to explain our main empirical findings.

2.3 Market structure - hedging with stocks and derivatives

In this section we expand the market defined in the previous section with a derivative. We define a derivative as an asset with a much smaller parameter uncertainty than the long-short portfolio. It has a payoff X_D and price P_D and can also be used to hedge the exogenous risk. Let $m_D \equiv E\left(X_D\right)$ and $b_D \equiv \frac{cov\left(X_Q, X_D\right)}{Var(X_D)}$. We assume that all agents aobserve the non-zero correlation between the payoffs of the long-short portfolio and the derivative. We refer to this new setting case as the "extended" case. As before, we assume that the payoffs and the parameters in the economy follow a multivariate Normal distributions

$$\begin{pmatrix} X_M \\ X_H \\ X_D \\ X_Q \end{pmatrix} \sim_{|m_H,m_D,b_H,b_D} N \left[\begin{pmatrix} \mu_M \\ m_H \\ m_D \\ \mu_Q \end{pmatrix}, \begin{pmatrix} \sigma_M^2 & 0 & 0 & 0 \\ 0 & \sigma_H^2 & \beta_{D,H}\sigma_H^2 & b_H\sigma_H^2 \\ 0 & \beta_{H,D}\sigma_D^2 & \sigma_D^2 & b_D\sigma_D^2 \\ 0 & b_H\sigma_H^2 & b_D\sigma_D^2 & \sigma_Q^2 \end{pmatrix} \right], \tag{4}$$

$$\begin{pmatrix} m_H \\ m_D \\ b_H \\ b_D \end{pmatrix} \sim N \left[\begin{pmatrix} \eta_H \\ \eta_D \\ \beta_H \\ \beta_D \end{pmatrix}, \begin{pmatrix} \xi_{m_H}^2 & 0 & 0 & 0 \\ 0 & \xi_{m_D}^2 & 0 & 0 \\ 0 & 0 & \xi_{b_H}^2 & 0 \\ 0 & 0 & 0 & \xi_{b_D}^2 \end{pmatrix} \right], \tag{5}$$

with $cov\left(X_H, X_D\right) = \beta_{D,H}\sigma_H^2 = \beta_{H,D}\sigma_D^2$ and $r_{H,D} = \beta_{H,D}\beta_{D,H}$. First we solve the general case where there is parameter uncertainty in both the hedge long-short portfolio and the derivative. Later we assume a special case where $\xi_{b_D} = 0$ and show the effect of introducing a derivative without beta uncertainty into the economy.

2.4 Agents

Our setting consists of two (representative and price taking) agents which we index by $i \in \{I, U\}$. Both agents have smooth ambiguity aversion preferences as defined in (1) with risk-aversion γ, ambiguity aversion δ and initial wealth $W_0^{(i)} > 0$. Each trader knows her own type and knows about the existence of the other agent. Both agents are rational in a sense that they know how to rank alternatives and choose the one that maximizes her utility as in (1). And both of them are "smart" because they incorporate all the available information.

The agents are different across two dimensions. The first dimension is the information that the agent is endowed with. We label the first agent "Informed" (I) adn assume that she knows the draw of parameters m_H, m_d, b_H and b_D. This agent is effectively not ambiguous[5]. Let $\theta \equiv$

[5]More formally, for this agent, the distribution of parameters is a degenerate one at the true parameter.

$(m_H, b_H), P \equiv (P_M, P_H)$ for the restricted setting and $\theta \equiv (m_H, m_d, b_H, b_D), P \equiv (P_M, P_H, P_D)$ for the extended setting. The value function can then be written as:

$$V^{(I)}(W) = E(W|\theta) - \frac{\gamma}{2} Var(W|\theta). \tag{6}$$

We label the other agent "Uninformed" (U) because she does not observe the mean and expected payoff of the hedge asset. However the agent is aware of the existence of the informed agent, and knows that the informed agent observes the expected payoff and beta of the hedge asset. This crucial assumption implies that the equilibrium price conveys important information for the uninformed agent. As a consequence, the uninformed agent does not maximize utility using the distribution in (3) but takes it as a prior which is updated using the price vector. This leads to a posterior distribution with a density $f(\theta|P)$. Let $d\theta \equiv dm_H db_H$ for the restricted setting, and $d\theta \equiv dm_H dm_D db_H db_D$ for the extended one. The value function then becomes

$$\begin{aligned} V^{(U)}(W) &= \int E(W|\theta, P) f(\theta|P) d\theta - \frac{\gamma}{2} \int Var(W|\theta, P) f(\theta|P) d\theta \\ &\quad - \frac{\delta}{2} \int \left(E(W|\theta, P) - \int E(W|\theta, P) f(\theta|P) d\theta \right)^2 f(\theta|P) d\theta. \end{aligned} \tag{7}$$

Learning from prices affects the uncertainty level of the uninformed investor when assessing parameter uncertainty. We will discuss the way that the uninformed investor extracts information more formally in Section 3.2. Let $W^{(i)}$ be the terminal wealth of agent $i \in \{I, U\}$. We represent a portfolio by $\alpha_M^{(i)}, \alpha_F^{(i)}, \alpha_H^{(i)}, \alpha_D^{(i)}$. The terminal wealth for agent i is

$$W^{(i)} = R_F \alpha_F^{(i)} + X_M \alpha_M^{(i)} + X_Q q^{(i)} + X_H \alpha_H^{(i)} + X_D \alpha_D^{(i)}, \tag{8}$$

where $q^{(i)} \geq 0$ is the exposure of agent i to the exogenous risk. In the restricted setting, we set $\alpha_D^{(i)} \equiv 0$.

In addition to the information difference between agents, we also assume that informed and uninformed agents have a different exposure to the exogenous shock $q^{(i)}$. Again, we first solve the general case in which both of agents have a non-zero exposure. And later we assume that the uninformed agent is not exposed to the exogenous risk $q^{(U)} = 0$.

We think of this setting as a market with an institutional investor (bank or pension fund) and a hedge fund. On the one hand, the institutional investor has a desire to hedge the exogenous shock, for example to meet risk management or regulatory constraints. It has therefore invested in information technologies and is consequently better informed about the expected returns and hedging potential of the long-short portfolio. On the other hand, the hedge fund has not invested in information technologies because it is less constrained. The hedge fund learns about hedging potential and expected returns of the hedge asset through market prices. The hedge fund is willing to hedge the institutional investor's risk, if the risk premium is sufficiently large.

3 Hedging risk with only a long-short portfolio

In this section we solve for investor demands, equilibrium prices and risk premiums in the restricted setting (without a derivative) that is described in Section 2.2. We study the effect of beta uncertainty on the equilibrium risk premium.

3.1 Portfolio choice

In this subsection we define the optimization problem for both types of investor and solve for optimal demand functions.

3.1.1 Demand function for the informed investor

The informed investor's terminal wealth is given in (8) and upon setting $\alpha_D^{(I)} = 0$ becomes

$$W^{(I)} = R_F \alpha_F^{(I)} + X_M \alpha_M^{(I)} + X_Q q^{(I)} + X_H \alpha_H^{(I)}, \tag{9}$$

with $\alpha_F^{(I)}$, $\alpha_M^{(I)}$ and $\alpha_H^{(I)}$ respectively denoting the informed investor's demand for the risk-free, market and long-short portfolio and $q^{(I)}$ is the exposure to the risk factor X_Q. She faces the following budget constraint

$$W_0^{(I)} = \alpha_F^{(I)} + P_M \alpha_M^{(I)} + P_H \alpha_H^{(I)}, \tag{10}$$

with P_M and P_H denoting the price of the market and hedge assets. Through substitution of $\alpha_F^{(I)}$ we obtain

$$W^{(I)} = R_F W_0^{(I)} + X_Q q^{(I)} + \left(X_M - R_F P_M\right) \alpha_M^{(I)} + \left(X_H - R_F P_H\right) \alpha_H^{(I)}. \quad (11)$$

The informed investor's utility is given in (6), she knows m_H and b_H and thus solves

$$max_{\alpha_M^{(I)},\alpha_H^{(I)}} E\left(W^{(I)}|m_H, b_H\right) - \frac{\gamma}{2} Var\left(W^{(I)}|m_H, b_H\right), \quad (12)$$

where

$$E\left(W^{(I)}|m_H, b_H\right) = R_F W_0^{(I)} + \mu_Q q^{(I)} + \left(\mu_M - R_F P_M\right) \alpha_M^{(I)} + \left(m_H - R_F P_H\right) \alpha_H^{(I)} \quad (13)$$

and

$$Var\left(W^{(I)}|m_H, b_H\right) = \sigma_Q^2 {q^{(I)}}^2 + \sigma_M^2 {\alpha_M^{(I)}}^2 + \sigma_H^2 {\alpha_H^{(I)}}^2 + 2 b_H \sigma_H^2 q^{(I)} \alpha_H^{(I)}. \quad (14)$$

The first-order conditions of (12) with respect to $\alpha_M^{(I)}$ and $\alpha_H^{(I)}$ are

$$\begin{aligned} \left(\mu_M - R_F P_M\right) - \gamma \alpha_M^{(I)} \sigma_M^2 &= 0 \quad (15) \\ \left(m_H - R_F P_H\right) - \gamma \left(\alpha_H^{(I)} \sigma_H^2 + b_H \sigma_H^2 q^{(I)}\right) &= 0. \quad (16) \end{aligned}$$

By rearranging terms, we get the optimal demand functions for the market and the long-short portfolio:

$$\begin{aligned} \alpha_M^{(I)} &= \frac{\mu_M - R_F P_M}{\gamma \sigma_M^2}, \quad (17) \\ \alpha_H^{(I)} &= \frac{m_H - R_F P_H}{\gamma \sigma_H^2} - b_H q^{(I)}. \quad (18) \end{aligned}$$

The demand of the long-short portfolio consists of two terms. The first is the speculative demand which is increasing in profitability and decreasing in risk. The second term is the hedge demand which is the product of the beta (hedging ability) and exogenous risk exposure. Since the market portfolio is assumed to be independent of the exogenous risk and the long-short portfolio, the demand for the market portfolio has only a speculative part.

3.1.2 Demand function for the uninformed investor

The uninformed investor's wealth is similar to that of the informed one in (8)

$$W^{(U)} = R_F W_0^{(U)} + X_Q q^{(U)} + (X_M - R_F P_M)\,\alpha_M^{(U)} + (X_H - R_F P_H)\,\alpha_H^{(U)}. \tag{19}$$

However, the uninformed investor does not observe m_H and b_H but takes into an account the existence of an informed agent. Let $\theta = (m_H, b_H)$, $d\theta = dm_H db_H$, $P = (P_M, P_H)$ and $f(\theta|P)$ be the posterior pdf of the parameter distribution. The utility of the uninformed investor is (7) and the maximization problem is

$$\begin{aligned} max_{\alpha_M^{(U)},\alpha_H^{(U)}} \quad & \int E\left(W^{(U)}|\theta, P\right) f(\theta|P)\, d\theta - \frac{\gamma}{2}\int Var\left(W^{(U)}|\theta, P\right) f(\theta|P)\, d\theta \\ & - \frac{\delta}{2}\int \left(E\left(W^{(U)}|\theta, P\right) - \int E\left(W^{(U)}|\theta, P\right) f(\theta|P)\, d\theta\right)^2 f(\theta|P)\, d\theta, \end{aligned} \tag{20}$$

where

$$\begin{aligned} & \int E\left(W^{(U)}|\theta, P\right) f(\theta|P)\, d\theta \\ & = R_F W_0^{(U)} + \mu_Q q^{(U)} + (\mu_M - R_F P_M)\,\alpha_M^{(U)} \\ & + \left(\int E(X_H|\theta, P) f(\theta|P)\, d\theta - R_F P_H\right)\alpha_H^{(U)} \\ & = R_F W_0^{(U)} + \mu_Q q^{(U)} + (\mu_M - R_F P_M)\,\alpha_M^{(U)} + (E(m_H|P) - R_F P_H)\,\alpha_H^{(U)}, \end{aligned} \tag{21}$$

$$\begin{aligned} & \int Var\left(W^{(U)}|\theta, P\right) f(\theta|P)\, d\theta \\ & = \sigma_Q^2 {q^{(U)}}^2 + \sigma_M^2 {\alpha_M^{(U)}}^2 + \sigma_H^2 {\alpha_H^{(U)}}^2 + 2\int cov(X_Q, X_H|\theta, P) f(\theta|P)\, d\theta \alpha_H^{(U)} q^{(U)} \\ & = \sigma_Q^2 {q^{(U)}}^2 + \sigma_M^2 {\alpha_M^{(U)}}^2 + \sigma_H^2 {\alpha^{(U)}}^2 + 2E(b_H|P)\,\sigma_H^2 \alpha_H^{(U)} q^{(U)} \end{aligned} \tag{22}$$

and

$$\int \left(E\left(W^{(U)}|\theta,P\right) - \int E\left(W^{(U)}|\theta,P\right) f\left(\theta|P\right) d\theta \right)^2 f\left(\theta|P\right) d\theta$$
$$= \int \left(m_H - E\left(m_H|P\right)\right)^2 f\left(\theta|P\right) d\theta \alpha_H^{(I)^2}$$
$$= Var\left(m_H|P\right) \alpha_H^{(I)^2}. \tag{23}$$

The first order conditions of (20) with respect to $\alpha_M^{(U)}$ and $\alpha_H^{(U)}$ are

$$\left(\mu_M - R_F P_M\right) - \gamma \alpha_M^{(U)} \sigma_M^2 = 0 \tag{24}$$
$$\left(E\left(m_H|P\right) - R_F P_H\right) - \gamma \left(\sigma_H^2 \alpha_H^{(U)} + E\left(b_H|P\right) \sigma_H^2 q^{(U)}\right) - \delta Var\left(m_H|P\right) \alpha_H^{(U)} = 0. \tag{25}$$

By rearranging terms, we can get the uninformed agent's optimal demand function for both assets:

$$\alpha_M^{(U)} = \frac{\mu_M - R_F P_M}{\gamma \sigma_M^2} \tag{26}$$
$$\alpha_H^{(U)} = \frac{E\left(m_H|P\right) - R_F P}{\gamma \sigma_H^2 + \delta Var\left(m_H|P\right)} - E\left(b_H|P\right) q^{(U)} \frac{\gamma \sigma_H^2}{\gamma \sigma_H^2 + \delta Var\left(m_H|P\right)}. \tag{27}$$

The uninformed investor's demand functions are thus very similar to the one of the informed investor in (18). Speculative demand is increasing in expected returns and decreasing in variance, risk and ambiguity aversion. However, the uninformed investor does not observe m and therefore takes an expectation. As before, uncertainty consists of two components (i) payoff risk reflected in $\gamma\sigma_H^2$ and (ii) parameter uncertainty in $\delta Var\left(m_H|P\right)$. Similar to the informed investor, hedging demand is also a function of *expected* beta $E\left(b_H|P\right)$, shock exposure $q^{(U)}$ and it again decreases in parameter uncertainty $Var\left(m_H|P\right)$. More generally, parameter uncertainty shrinks demands to zero, conditional on $E\left(m_H|P\right)$ and $E\left(b_H|P\right)$. As a consequence, the uninformed investor's demand will be lower (in absolute terms) than informed investor's demand. It is important to note that beta uncertainty ξ_b^2 does not affect demands directly, but only through ambiguity about expected payoffs $Var\left(m_H|P\right)$[6]. This is likely an important reason why beta uncertainty received

[6]We show this more formally in equation (33).

less attention in the literature than the uncertainty about mean returns. The uninformed investor's demand in equaton (27) seems not unaffected by beta uncertainty. However, we will show in the next section, which discusses learning from market prices, that this observation is not true.

3.2 Learning through prices

Equation (27) implicitly reflects the idea that prices convey important information for uninformed investors. We assume that the uninformed investor applies Bayes rule to form a posterior belief about m_H and b_H. To do that we assume that the uninformed agents know: (i) that the other type observes m_H and b_H and (ii) the distributions from which m_H and b_H are drawn. Moreover, an uninformed agent can derive optimal demand functions (18) and (27) and thus knows the inputs used by the auctioneer to find prices that clear markets. In other words, the uninformed investor knows that total demand equals total supply for the observed price level. And since we assume that the hedge asset is in zero net supply

$$\alpha_H^{(I)} + \alpha_H^{(U)} = 0. \tag{28}$$

If we substitute the informed agent's optimal demand (18) we can express prices as a function of known parameters ($\alpha_H^{(U)}$, σ_H^2, $q^{(I)}$ and R_F) and two independent draws from a normal distribution (m_H and b_H)

$$P_H = \frac{1}{R_F} m_H - \frac{\gamma \sigma_H^2 q^{(I)}}{R_F} b_H + \alpha^{(U)} \frac{\gamma \sigma_H^2}{R_F}. \tag{29}$$

Put differently, the uninformed agent observes that prices are high (low) and consequently learns that informed investor's demand is high (low). But remains in doubt as to whether that is related to hedging (b_H) or speculative motives (m_H). Formally, the uninformed agent uses the prior distribution in (3) together with the relationship between the observed price and the parameters given in (29) and produces a posterior distribution by applying Bayes law. Since P_H is a linear combination of the uncertain parameters which are in turn normally distributed, the posterior has the following form:

$$\begin{pmatrix} m_H \\ b_H \end{pmatrix} \sim_{|P} N \left[\begin{pmatrix} E(m_H|P) \\ E(b_H|P) \end{pmatrix}, \begin{pmatrix} Var(m_H|P) & cov(m_H, b_H|P) \\ cov(m_H, b_H|P) & Var(b_H|P) \end{pmatrix} \right] \tag{30}$$

where

$$E(m_H|P) = \eta_H + R_F \frac{\xi_{m_H}^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2} \left[P_H - \left(\frac{1}{R_F}\eta_H - \frac{\gamma\sigma_H^2 q^{(I)}}{R_F}\beta_H + \alpha_H^{(U)} \frac{\gamma\sigma_H^2}{R_F} \right) \right] \tag{31}$$

$$E(b_H|P) = \beta_H + R_F \frac{\gamma\sigma_H^2 q^{(I)} \xi_{b_H}^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2} \left[P_H - \left(\frac{1}{R_F}\eta_H - \frac{\gamma\sigma_H^2 q^{(I)}}{R_F}\beta_H + \alpha_H^{(U)} \frac{\gamma\sigma_H^2}{R_F} \right) \right] \tag{32}$$

and

$$Var(m_H|P) = \xi_{m_H}^2 - \frac{\left(\xi_{m_H}^2\right)^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2} \tag{33}$$

$$Var(b_H|P) = \xi_{b_H}^2 - \frac{\left(\gamma\sigma_H^2 q^{(I)} \xi_{b_H}^2\right)^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2} \tag{34}$$

$$cov(m_H, b_H|P) = \frac{\left(\gamma\sigma_H^2 q^{(I)}\right) \xi_{m_H}^2 \xi_{b_H}^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2}. \tag{35}$$

Although the investor has the ability to learn from the entire price vector P, only the price of the hedge asset P_H actually conveys information. The price of the market is uninformative because market payoff is independent of all the other payoffs in the economy.

The main goal of our paper is to study the effect of beta uncertainty on hedging demands and risk premiums for the stock and derivative. A simple - albeit imperfect - way to accomplish this goal would be through the uninformed investor's demand function (27). We know that the uninformed agent's demand decreases in $Var(m_H|P)$ - if we keep $E(m_H|P)$ and $E(b_H|P)$ constant. It is straightforward to see that $Var(m_H|P)$ increases in $\xi_{b_H}^2$

$$\frac{\partial Var(m_H|P)}{\partial \xi_{b_H}^2} = \left(\frac{\gamma\sigma_H^2 q^{(I)} \xi_{m_H}^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2 \xi_b^2} \right)^2 > 0 \tag{36}$$

and thus, in equilibrium, uninformed investors dislike hedging uncertainty and therefore decrease their demands.

3.3 Equilibrium

In this subsection we derive equilibrium demands and price functions for both types of agents. We also show the existence of an equilibrium, while we abstract from the process that leads to the equilibrium. The equilibrium is defined as follows:

Definition 3.1 *An equilibrium is represented by prices* P_H^*, P_M^* *and demands* $\alpha_H^{(I)*}, \alpha_M^{(I)*}$, $\alpha_H^{(U)*}$ *and* $\alpha_M^{(U)*}$ *such that (i) each investor maximizes utility of future wealth expressed in* (1)*, (ii) uninformed investors update their beliefs about* m_H *and* b_H *and (iii) markets clear.*

Proposition 3.2 *Given the assumptions of the restricted setting that is described in Section 2.2 and given a realization of* m_H *and* b_H*, there is a unique set of prices* P_H^*, P_M^* *and demands* $\alpha_H^{(I)*}, \alpha_M^{(I)*}$, $\alpha_H^{(U)*}$ *and* $\alpha_M^{(U)*}$ *that solves the equilibrium condition. These prices are given as*

$$P_M^* = \frac{1}{R_F}\left(\mu_M - \frac{1}{2}\gamma\sigma_M^2\overline{\alpha}_M\right) \tag{37}$$

and

$$\begin{aligned} P_H^* &= \frac{1}{R_F}\left\{\frac{\gamma\sigma_H^2\left[m_H + E\left(m_H|P^*\right)\right] + \delta Var\left(m_H|P^*\right)m_H}{2\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)}\right. \\ &\quad \left. - \frac{\gamma\sigma_H^2\left[\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)\right]}{2\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)}\left[b_H q^{(I)} + E\left(b_H|P^*\right)q^{(U)}\frac{\gamma\sigma_H^2}{\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)}\right]\right\}, \end{aligned} \tag{38}$$

with

$$E\left(m_H|P^*\right) = \eta_H + \frac{\xi_{m_H}^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2\xi_{b_H}^2}\left[\left(m_H - \eta_H\right) - \gamma\sigma_H^2 q^{(I)}\left(b_H - \beta_H\right)\right] \tag{39}$$

$$Var\left(m_H|P^*\right) = \xi_{m_H}^2 - \frac{\left(\xi_{m_H}^2\right)^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2\xi_{b_H}^2} \tag{40}$$

$$E\left(b_H|P^*\right) = \beta_H + \frac{\gamma\sigma_H^2 q^{(I)}\xi_{b_H}^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2\xi_{b_H}^2}\left[\left(m_H - \eta_H\right) - \gamma\sigma_H^2 q^{(I)}\left(b_H - \beta_H\right)\right]. \tag{41}$$

where $\overline{\alpha}_M$ *is the total supply of the market portfolio. The demands are given by plugging* (37)-(41) *into* (17)*,* (18)*,* (26) *and* (27)*.*

It is straightforward to see that equilibrium prices and demands are thus affected by beta uncertainty. In fact, our model collapses to a standard mean-variance framework if we eliminate beta uncertainty.

Proposition 3.3 *In equilibrium,* $\xi_{b_H}^2 = 0$ *implies that* $Var(m_H|P^*) = 0$ *The equilibrium then reduces to a mean-variance type without uncertainty.*

This seemingly trivial proposition has important implications. It shows that uninformed investors care about beta uncertainty, even while such uncertainty does not directly feed into their utility function. Beta uncertainty ($\xi_{b_H}^2$) is important because it enables uninformed investors to learn about the risky asset's expected payoffs m_H which affects their utility. It is also important to note that informational differences between agents disappear if we take hedging uncertainty out of the equation ($\xi_{b_H}^2$=0). In that case, prices are a linear function of expected payoffs and uninformed investors observe m_H (without any uncertainty) through prices.

3.3.1 Equilibrium risk premium

As a final step in the setting without the derivative, we study the relation between beta uncertainty and the risk premium. We in this section that the risk premium is a function of the initial parameter draw (m_H, b_H). Hence, we need to make the simplifying assumption that m_H and b_H equal their unconditional means $m_H = \eta_H$, $b_H = \beta_H$. By using the updating rules, we can then derive that conditional expectations will be equal to the unconditional ones, $E(m_H) = E(m_H|P) = \eta_H$, $E(b_H) = E(b_H|P) = \beta_H$. This was done to simplify our analysis. Since the price of the asset depends on the realization of the uncertain parameters, the premium will be different for every draw of the parameters. Thus in order to make a prediction we need to set the parameters to a specific level which we chose to be the unconditional expectation. We have pointed out in the previous subsection that beta uncertainty does not affect demands and prices directly, but only through $Var(m_H|P^*)$. More beta uncertainty thus leads to larger uncertainty about expected payoffs and lower demands for the uninformed agent. In this section we study the effect of changes in $Var(m|P_H^*)$ on the risk premium. Because prices are positive, we can - without loss of generality - focus on the numerator of the risk premium ($\frac{\eta_H - R_F P_H^*}{P_H^*}$)

$$\eta_H - R_F P_H^* = \frac{\gamma\sigma_H^2\left[\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)\right]}{2\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)}\left(\beta_H\left[q^{(I)} + q^{(U)}\frac{\gamma\sigma_H^2}{\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)}\right]\right). \quad (42)$$

If we take the derivative with respect to $Var(m|P^*)$:

$$\frac{\partial\left(\eta_H - R_F P_H^*\right)}{\partial Var\left(m_H|P^*\right)} = \frac{\gamma^2\delta\left(\sigma_H^2\right)^2}{\left[2\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)\right]^2}\left[\beta_H\left(q^{(I)} - q^{(U)}\right)\right]. \quad (43)$$

Based on these two equations we can write the following theorem:

Theorem 3.4 *If we assume that $q^{(I)}, q^{(I)} > 0$, then the total premium that the long-short portfolio carries has the same sign as the beta. If we add the assumption that $q^{(I)} > q^{(U)}$, then the total premium is an increasing function (in absolute terms) of the uncertainty level.*

Equation (42) shows that the sign of the risk premium equals the sign of β_H, if we assume that both agents are positively exposed to the shock $q^{(I)} > q^{(U)} \geq 0$. For example, if both agents are positively exposed to a negative shock and the payoff of the hedge asset is negatively correlated with the shock (i.e. the hedge asset provides insurance), we observe a negative premium.

Equation (43) shows that the premium is increasing in parameter uncertainty (in absolute terms). More specifically, the sensitivity of the risk premium with respect to beta uncertainty is a function of two factors. The first term in (42) can be seen as a generalization of the market risk-aversion, which incorporates both risk and ambiguity aversion and is increasing in beta uncertainty. The second term represents the sum of the respective hedging demands, where the uninformed investor's demand decreases in beta uncertainty. Which of these forces dominates depends on the difference in exposures and expected beta.[7] The finding can be explained as follows. An informed

[7]We also note that $Var\left(m_H|P_H^*\right) \to \infty$ implies

$$\eta_H - R_F P_H^* = \gamma\sigma^2\beta_H q^{(I)},$$

which means that the uninformed investor is driven out of the market if the hedging uncertainty becomes large enough.

investor is more exposed and wants to sell (buy) the asset to hedge her exposure. Since the asset is in zero net supply, the uninformed investor must be convinced to buy (sell) the asset. However, the uninformed investor is ambiguity averse and does not know the true beta and expected payoffs and therefore demands a higher risk premium to buy (sell) the asset.

4 Expanding the investment universe with a derivative

We motivate our study from an investor's perspective and want to quantify the idea that institutional investors who want to hedge inflation or crash risk prefer TIPS or options over long-short equity portfolios. Put differently, we want to test whether they prefer assets with less beta uncertainty over ones with more beta uncertainty. We therefore expand the investment universe with a derivative. While trading the derivative the agent faces much less parameter uncertainty. Thus we assume a market structure that is defined in Section 2.3. As before, we solve for prices, premiums and study the change in the premium of a long-short portfolio when the derivative is introduced. We find that an inclusion of a derivative results in a demand shift from the long-short portfolio to the derivative. This shift leads to a reduction in the total premium of the long-short portfolio. The degree of premium reduction depends on the correlation between the secure and uncertain asset.

4.1 Portfolio choice

In this section we define the optimization problem of the two agents in the extended setting. Using the FOC we derive the demand of the two agents.

4.1.1 Informed investor

As before, by substitution of the budget constraint into the terminal wealth we can get the following expression for the terminal wealth of the informed investor

$$W^{(I)} = R_F W_0^{(I)} + X_Q q^{(I)} + (X_M - R_F P_M)\,\alpha_M^{(I)} + (X_H - R_F P_H)\,\alpha_H^{(I)} + (X_D - R_F P_D)\,\alpha_D^{(I)}. \quad (44)$$

Similar to the restricted case

$$max_{\alpha_M^{(I)},\alpha_H^{(I)},\alpha_D^{(I)}} E\left(W^{(I)}|m_H, m_D, b_H, b_D\right) - \frac{\gamma}{2} Var\left(W^{(I)}|m_H, m_D, b_H, b_D\right), \tag{45}$$

where

$$\begin{aligned} & E\left(W^{(I)}|m_H, m_D, b_H, b_D\right) \\ & = R_F W_0^{(I)} + \mu_Q q^{(I)} + (\mu_M - R_F P_M)\, \alpha_M^{(I)} + (m_H - R_F P_H)\, \alpha_H^{(I)} + (m_D - R_F P_D)\, \alpha_D^{(I)}, \end{aligned} \tag{46}$$

and

$$\begin{aligned} & Var\left(W^{(I)}|m_H, m_D, b_H, b_D\right) \\ & = \sigma_Q^2 {q^{(I)}}^2 + \sigma_M^2 {\alpha_M^{(I)}}^2 + \sigma_H^2 {\alpha_H^{(I)}}^2 + \sigma_D^2 {\alpha_D^{(I)}}^2 \\ & + 2b_H \sigma_H^2 q^{(I)} \alpha_H^{(I)} + 2b_D \sigma_D^2 q^{(I)} \alpha_D^{(I)} + 2cov\left(X_H, X_D\right) \alpha_H^{(I)} \alpha_D^{(I)}. \end{aligned} \tag{47}$$

The first order conditions of (45) with respect to $\alpha_M^{(I)}$, $\alpha_H^{(I)}$ and $\alpha_D^{(I)}$ are

$$(\mu_M - R_F P_M) - \gamma \sigma_M^2 \alpha_M^{(I)} = 0 \tag{48}$$

$$(m_H - R_F P_H) - \gamma\left(\sigma_H^2 \alpha_H^{(I)} + b_H \sigma_H^2 q^{(I)} + \beta_{D,H} \sigma_H^2 \alpha_D^{(I)}\right) = 0 \tag{49}$$

$$(m_D - R_F P_D) - \gamma\left(\sigma_D^2 \alpha_D^{(I)} + b_D \sigma_D^2 q^{(I)} + \beta_{H,D} \sigma_D^2 \alpha_H^{(I)}\right) = 0, \tag{50}$$

By rearranging terms, we get the optimal demand function for the market, long-short portfolio and the derivative:

$$\alpha_M^{(I)} = \frac{\mu_M - R_F P_M}{\gamma \sigma_M^2} \tag{51}$$

$$\alpha_H^{(I)} = \frac{\frac{m_H - R_F P_H}{\gamma \sigma_H^2} - b_H q^{(I)} - \beta_{D,H}\left(\frac{m_D - R_F P_D}{\gamma \sigma_D^2} - b_D q^{(I)}\right)}{1 - r_{H,D}^2} \tag{52}$$

$$\alpha_D^{(I)} = \frac{\frac{m_D - R_F P_D}{\gamma \sigma_D^2} - b_D q^{(I)} - \beta_{H,D}\left(\frac{m_H - R_F P_H}{\gamma \sigma_H^2} - b_H q^{(I)}\right)}{1 - r_{H,D}^2}. \tag{53}$$

The interpretation of the demand functions is very similar to the one asset setting. In fact, the first two terms in the numerator of (52) and (53) are equivalent to equation (18) and thus consist of the standard speculative and hedging demands. However, the second part of the numerator takes into account the speculative and hedging potential for the other asset. For example, if the derivative has a very good risk-return trade-off and hedging potential, this affects the demand for the risky asset in a negative way. Likewise, the demand for the derivative depends negatively on the long-short portfolio's risk-return trade-off and hedging potential. This interpretation is quite intuitive as each investor allocates more wealth to the asset that better matches the speculative and hedging needs. It is also important to note that these substitution effects across assets emerge even if the assets are almost identical[8] ($m_H = m_D$, $\sigma_H^2 = \sigma_D^2$ and $b_H = b_D$) because the second asset offers diversification benefits ($\beta_{H,D} \neq 0$, $r_{H,D} \neq 0$).

4.1.2 Uninformed investor

The expression for the uninformed investor's wealth is similar to that of the informed one

$$W^{(U)} = R_F W_0^{(U)} + X_Q q^{(U)} + (X_M - R_F P_M)\,\alpha_M^{(U)} + (X_H - R_F P_H)\,\alpha_H^{(U)} + (X_D - R_F P_D)\,\alpha_D^{(U)}. \quad (54)$$

And as before, the uninformed investor does not observe the parameters but is allowed to learn from prices. Let $\theta = (m_H, m_D, b_H, b_D)$, $d\theta = dm_H dm_D db_H db_D$ and $P = (P_M, P_H, P_D)$. The maximization problem that the uninformed investor solves is

$$\begin{aligned} max_{\alpha_M^{(U)},\alpha_H^{(U)},\alpha_D^{(U)}} \quad & \int E\left(W^{(U)}|\theta,P\right) f\left(\theta|P\right) d\theta - \frac{\gamma}{2}\int Var\left(W^{(U)}|\theta,P\right) f\left(\theta|P\right) d\theta \\ & -\frac{\delta}{2}\int\left(E\left(W^{(U)}|\theta,P\right) - \int E\left(W^{(U)}|\theta,P\right) f\left(\theta|P\right) d\theta\right)^2 f\left(\theta|P\right) d\theta. \quad (55)\end{aligned}$$

[8] Note that the assets are not identical because the correlation uncertainty of the derivative is much lower than that of the risky asset.

where

$$\int E\left(W^{(U)}|\theta,P\right)f\left(\theta|P\right)d\theta$$
$$=R_F W_0^{(U)}+\mu_Q q^{(U)}+\left(\mu_M-R_F P_M\right)\alpha_M^{(U)}$$
$$+\left(E\left(m_H|P\right)-R_F P_H\right)\alpha_H^{(U)}+\left(E\left(m_D|P\right)-R_F P_D\right)\alpha_D^{(U)}, \tag{56}$$

$$\int Var\left(W^{(U)}|\theta,P\right)f\left(\theta|P\right)d\theta$$
$$=\sigma_Q^2 {q^{(U)}}^2+\sigma_M^2 {\alpha_M^{(U)}}^2+\sigma_H^2 {\alpha^{(U)}}^2+\sigma_D^2 {\alpha^{(U)}}^2$$
$$+2E\left(b_H|P\right)\sigma_H^2\alpha_H^{(U)}q^{(U)}+2E\left(b_D|P\right)\sigma_D^2\alpha_D^{(U)}q^{(U)}+2cov\left(X_H,X_D\right)\alpha_H^{(U)}\alpha_D^{(U)} \tag{57}$$

and

$$\int\left(E\left(W^{(U)}|\theta,P\right)-\int E\left(W^{(U)}|\theta,P\right)f\left(\theta|P\right)d\theta\right)^2 f\left(\theta|P\right)d\theta$$
$$=Var\left(m_H|P\right){\alpha_H^{(I)}}^2+Var\left(m_D|P\right){\alpha_D^{(I)}}^2 \tag{58}$$

The first order conditions of (55) with respect to $\alpha_M^{(U)}$, $\alpha_H^{(U)}$ and $\alpha_D^{(U)}$ are

$$\left(\mu_M-R_F P_M\right)-\gamma\sigma_M^2\alpha_M^{(U)}=0 \tag{59}$$

$$\left(E\left(m_H|P\right)-R_F P_H\right)-\gamma\left(\sigma_H^2\alpha_H^{(U)}+E\left(b_H|P\right)\sigma_H^2 q^{(U)}+\beta_{D,H}\sigma_H^2\alpha_D^{(U)}\right)$$
$$-\delta\left(Var\left(m_H|P\right)\alpha_H^{(U)}+cov\left(m_H,m_D|P\right)\alpha_D^{(U)}\right)=0 \tag{60}$$

$$\left(E\left(m_D|P\right)-R_F P_D\right)-\gamma\left(\sigma_D^2\alpha_D^{(U)}+E\left(b_D|P\right)\sigma_D^2 q^{(U)}+\beta_{H,D}\sigma_D^2\alpha_H^{(U)}\right)$$
$$-\delta\left(Var\left(m_D|P\right)\alpha_D^{(U)}+cov\left(m_H,m_D|P\right)\alpha_H^{(U)}\right)=0. \tag{61}$$

By rearranging terms, we can get the uninformed agent's optimal demand functions:

$$\alpha_M^{(U)} = \frac{\mu_M - R_F P_M}{\gamma \sigma_M^2}. \tag{62}$$

$$\begin{aligned}\alpha_H^{(U)} = \frac{1}{1 - C_H C_D} \times \big(& B_H \left(E\left(m_H|P\right) - R_F P_H\right) - E\left(b_H|P\right) q^{(U)} A_H \\ & - C_H \left(B_D \left(E\left(m_D|P\right) - R_F P_D\right) - E\left(b_D|P\right) q^{(U)} A_D\right)\big)\end{aligned} \tag{63}$$

$$\begin{aligned}\alpha_D^{(U)} = \frac{1}{1 - C_H C_D} \times \big(& B_D \left(E\left(m_D|P\right) - R_F P_D\right) - E\left(b_D|P\right) q^{(U)} A_D \\ & - C_D \left(B_H \left(E\left(m_H|P\right) - R_F P_H\right) - E\left(b_H|P\right) q^{(U)} A_H\right)\big),\end{aligned} \tag{64}$$

where A_H, A_D, B_H, B_D, C_H and C_D are defined in Appendix A.3.

While the optimal demand functions may seem involved, the structure closely resembles that of the informed investor in (52) and (53) and the uninformed agent's optimal demand in the previous section (27). More specifically, the first term closely resembles the uninformed agent's demand in the one asset case in (27). It shows that demand for an asset increases in expected payoffs and expected hedging potential and decreases in risk. However, there are two differences. First, the agent observes three prices (P_M, P_H and P_D) and therefore learns using all three. Second, the agent faces more uncertain parameters (m_H, m_D, b_H, b_D). The expected benefits in the numerators are therefore scaled by more terms (capturing the additional parameter uncertainty) in the denominators.

Similar to the informed agent's demand function in (18), the second term represents substitution from one asset to the other. Substitution effects are stronger when (i) expected payoffs and expected hedging potential of the other asset are larger or (ii) the risks of the other asset are relatively small.[9]

[9] Note that A_H, A_D, B_H, B_D, C_H and C_D are positive (scaling) factors. Moreover we show in equation (78) that $cov(m_H, m_D|P) = 0$ and therefore that $C_H C_D < 1$.

4.2 Efficient learning through prices

We explain in this subsection how the uninformed investor learns about the unobserved parameters by observing a vector of prices $P = (P_M, P_H, P_D)$. We apply the same procedure as in section 3.2. It relies on the fact that agents know the utility function of the other agent, parameter distributions and the price forming mechanism followed by the auctioneer. If we substitute the informed agent's optimal demand functions as specified in (52) and (53) in the market clearance conditions we find the following prices for the long-short portfolio and the derivative

$$P_H = \frac{1}{R_F} m_H - \frac{\gamma \sigma_H^2 q^{(I)}}{R_F} b_H + \left(\alpha_H^{(U)} - \beta_{D,H} \alpha_D^{(U)}\right) \frac{\gamma \sigma_H^2}{R_F} \tag{65}$$

$$P_D = \frac{1}{R_F} m_D - \frac{\gamma \sigma_D^2 q^{(I)}}{R_F} b_D + \left(\alpha_D^{(U)} - \beta_{H,D} \alpha_H^{(U)}\right) \frac{\gamma \sigma_D^2}{R_F}. \tag{66}$$

These price functions (65) and (66), have a very similar structure as (29) - except for the intercept. More importantly, each asset's price function (65) and (66) depends only on parameters specific to that particular asset because $\beta_{H,D}$ is known. Again, we update expectations and parameter uncertainty regarding the unobserved parameters (m_H, m_D, b_H and b_D) using price information and multivariate normality as in section 3.2. The posterior distribution of the parameters is

$$\begin{pmatrix} m_H \\ m_D \\ b_H \\ b_D \end{pmatrix} \sim_{|P} N \left[\begin{pmatrix} E(m_H|P) \\ E(m_D|P) \\ E(b_H|P) \\ E(b_D|P) \end{pmatrix}, \begin{pmatrix} Var(m_H|P) & 0 & cov(m_H, b_H|P) & 0 \\ 0 & Var(m_D|P) & 0 & cov(m_D, b_D|P) \\ cov(m_H, b_H|P) & 0 & Var(b_H|P) & 0 \\ 0 & cov(m_D, b_D|P) & 0 & Var(b_D|P) \end{pmatrix} \right] \tag{67}$$

where

$$E(m_H|P) = \eta_H + R_F \frac{\xi_{m_H}^2}{\xi_{m_H}^2 + \left(\gamma \sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2} \times \left[P_H - \left(\frac{1}{R_F} \eta_H - \frac{\gamma \sigma_H^2 q^{(I)}}{R_F} \beta_H + \left(\alpha_H^{(U)} - \beta_{D,H} \alpha_D^{(U)}\right) \frac{\gamma \sigma_H^2}{R_F} \right) \right] \tag{68}$$

$$E\left(m_D|P\right)=\eta_D+R_F\frac{\xi_{m_D}^2}{\xi_{m_D}^2+\left(\gamma\sigma_D^2q^{(I)}\right)^2\xi_{b_D}^2}\times$$
$$\left[P_D-\left(\frac{1}{R_F}\eta_D-\frac{\gamma\sigma_D^2q^{(I)}}{R_F}\beta_D+\left(\alpha_D^{(U)}-\beta_{H,D}\alpha_H^{(U)}\right)\frac{\gamma\sigma_D^2}{R_F}\right)\right] \tag{69}$$

$$E\left(b_H|P\right)=\beta_H+R_F\frac{\gamma\sigma_H^2q^{(I)}\xi_{b_H}^2}{\xi_{m_H}^2+\left(\gamma\sigma_H^2q^{(I)}\right)^2\xi_{b_H}^2}\times$$
$$\left[P_H-\left(\frac{1}{R_F}\eta_H-\frac{\gamma\sigma_H^2q^{(I)}}{R_F}\beta_H+\left(\alpha_H^{(U)}-\beta_{D,H}\alpha_D^{(U)}\right)\frac{\gamma\sigma_H^2}{R_F}\right)\right] \tag{70}$$

$$E\left(b_D|P\right)=\beta_D+R_F\frac{\gamma\sigma_D^2q^{(I)}\xi_{b_D}^2}{\xi_{m_D}^2+\left(\gamma\sigma_D^2q^{(I)}\right)^2\xi_{b_D}^2}\times$$
$$\left[P_D-\left(\frac{1}{R_F}\eta_D-\frac{\gamma\sigma_D^2q^{(I)}}{R_F}\beta_D+\left(\alpha_D^{(U)}-\beta_{H,D}\alpha_H^{(U)}\right)\frac{\gamma\sigma_D^2}{R_F}\right)\right] \tag{71}$$

$$Var\left(m_H|P\right)=\xi_{m_H}^2-\frac{\left(\xi_{m_H}^2\right)^2}{\xi_{m_H}^2+\left(\gamma\sigma_H^2q^{(I)}\right)^2\xi_{b_H}^2} \tag{72}$$

$$Var\left(m_D|P\right)=\xi_{m_D}^2-\frac{\left(\xi_{m_D}^2\right)^2}{\xi_{m_D}^2+\left(\gamma\sigma_D^2q^{(I)}\right)^2\xi_{b_D}^2} \tag{73}$$

$$Var\left(b_H|P\right)=\xi_{b_H}^2-\frac{\left(\gamma\sigma_H^2q^{(I)}\xi_{b_H}^2\right)^2}{\xi_{m_H}^2+\left(\gamma\sigma_H^2q^{(I)}\right)^2\xi_b^2} \tag{74}$$

$$Var\left(b_D|P\right)=\xi_{b_D}^2-\frac{\left(\gamma\sigma_D^2q^{(I)}\xi_{b_D}^2\right)^2}{\xi_{m_D}^2+\left(\gamma\sigma_D^2q^{(I)}\right)^2\xi_D^2} \tag{75}$$

$$cov\left(m_H,b_H|P\right)=\frac{\left(\gamma\sigma_H^2q^{(I)}\right)\xi_{m_H}^2\xi_{b_H}^2}{\xi_{m_H}^2+\left(\gamma\sigma_H^2q^{(I)}\right)^2\xi_{b_H}^2} \tag{76}$$

$$cov\left(m_D,b_D|P\right)=\frac{\left(\gamma\sigma_D^2q^{(I)}\right)\xi_{m_D}^2\xi_{b_D}^2}{\xi_{m_D}^2+\left(\gamma\sigma_D^2q^{(I)}\right)^2\xi_{b_D}^2}. \tag{77}$$

An important conclusion is that

$$cov\left(m_H, m_D|P\right) = 0 \tag{78}$$

These learning rules define how the uninformed investor updates her expectations and uncertainty about the unobserved parameters in the most efficient way.

4.3 Equilibrium

Similar to the restricted case, we define an equilibrium in terms of prices and demand functions and abstract from the path towards the equilibrium.

Definition 4.1 *An equilibrium is represented by prices P_M^*, P_H^* and P_D^* and demands ${\alpha_M^{(I)}}^*$, $\alpha_H^{(I)*}$, $\alpha_D^{(I)*}$, $\alpha_M^{(U)}$ $\alpha_H^{(U)*}$ and $\alpha_D^{(U)*}$ such that (i) each investor maximizes utility of future wealth expressed in* (1)*, (ii) uninformed investors update their beliefs about m_H, m_D, b_H and b_D in the most efficient way and (iii) markets clear.*

Proposition 4.2 *There is a unique set of equilibrium prices P_M^*, P_H^*,P_D^* and demands ${\alpha_M^{(I)}}^*$, $\alpha_H^{(I)*}$, $\alpha_D^{(I)*}$, $\alpha_M^{(U)}$ $\alpha_H^{(U)*}$ and $\alpha_D^{(U)*}$ that solve the equilibrium conditions in the extended setting. The prices are expressed as follows*

$$P_M^* = \frac{1}{R_F}\left(\mu_M - \frac{1}{2}\gamma\sigma_M^2\overline{\alpha}\right) \tag{79}$$

$$P_H^* = \frac{F_H G_D - F_D K_H}{G_H G_D - K_H K_D} \tag{80}$$

$$P_D^* = \frac{F_D G_H - F_H K_D}{G_H G_D - K_H K_D}, \tag{81}$$

where F_H, F_D, G_H, G_D, K_H, K_D and the conditional moments are defined in appendix A.4. The equilibrium demands are given by plugging the prices into the demand functions (51), (52), (53), (62), (63) *and* (64)*.*

These equilibrium prices are complex functions of parameter uncertainty and it is not straightforward to interpret the effect of beta uncertainty on the price of either asset.

4.3.1 Equilibrium risk premium

In this subsection we derive the risk premium for both assets and assume again that $b_H = \beta_H$, $b_D = \beta_D$, $m_H = \eta_H$ and $m_D = \eta_D$. This assumption fixes the unconditional expectations, the conditional expectations, and the realizations of each uncertain parameter (separately) on the same value. Using the equilibrium condition, we can write the risk premium in term of payoffs in the following way

$$\eta_H - R_F P_H^* = \frac{\tilde{F_H}\tilde{G_D} - \tilde{F_D}\tilde{K_H}}{\tilde{G_H}\tilde{G_D} - \tilde{K_H}\tilde{K_D}} \tag{82}$$

$$\eta_D - R_F P_D^* = \frac{\tilde{F_D}\tilde{G_H} - \tilde{F_H}\tilde{K_D}}{\tilde{G_D}\tilde{G_H} - \tilde{K_D}\tilde{K_H}}. \tag{83}$$

where $\tilde{F_H}$, $\tilde{F_D}$,$\tilde{G_H}$, $\tilde{G_D}$, $\tilde{K_H}$, $\tilde{K_D}$ are defined in Appendix A.5

4.4 Change in risk premium

The equilibrium premiums for the restricted (42) and expanded environment (82) and (83) enable us to test the main conjecture of the model. Namely that the premium on the long-short portfolio decreases (in absolute terms) after the derivative has been introduced into the market. We therefore express the difference in the total premium of the long-short portfolio between the simple and expanded setting as follows

$$\Delta\left(\eta_H - R_F P_H^*\right) = \frac{\gamma^2\delta\sigma_H^2\sigma_D^2 V_H\left[\beta_{D,H}\beta_D(\delta V_H + 2\gamma\sigma_H^2) - 2r_{H,D}^2\beta_D\gamma\sigma_H^2\right]}{(\delta V_H + 2\gamma\sigma_H^2)\left[\delta^2 V_H V_D + 2\delta\gamma(\sigma_H^2 V_D + \sigma_D^2 V_H) + 4\gamma^2\left(1 - r_{H,D}^2\right)\sigma_H^2\sigma_D^2\right]}\left(q^{(I)} - q^{(U)}\right). \tag{84}$$

Where V_j is a short-hand notation for $Var\left(m_j|P^*\right)$ for $j \in \{H, D\}$.

Theorem 4.3 *Under the assumption that $q^{(I)} > q^{(U)} \geq 0$ and that both assets provide a hedge against the exogenous shock ($\beta_H,\ \beta_D < 0$) and the two assets are positively correlated with each other (which implies that $\beta_{D,H} > 0$).*

1. *If only the long-short portfolio is traded, and there is parameter uncertainty, there will be a*

negative total premium for holding the long-short portfolio.

2. *If we introduce a derivative, then for a high enough parameter uncertainty of the long-short portfolio, there will be a drop in the premium that is generated by the long-short portfolio (in absolute sense).*

3. *If there is no parameter uncertainty, the premium will not change when the derivative is introduced.*

These observations are the guidelines for our empirical exercise. Since uncertainty is closely related to what agents "perceive", it is extremely difficult to measure outside of a laboratory. Therefore we need a setting where we can test a prediction of such a model without obtaining a direct measure of the uncertainty level. The first two observations are the theoretical predictions that are generated by our model. And the last observation states that this phenomenon is unique to the setting of parameter uncertainty, and upon setting uncertainty to 0, the phenomenon disappears.

4.4.1 Numerical analysis

In this subsection we show that the effects documented in Theorem 4.3 are economically sizable under reasonable parameter choices. First we show the implication of observation (1) in Theorem 4.3. We calculate the equilibrium for a given set of parameters for the long-short portfolio:

[Table 1 about here.]

For simplicity of analysis we assume that γ and δ are equal across types and that the insurance provider (the hedge fund) is not exposed to the exogenous risk ($q^{(U)} = 0$). The choice for the risk-free rate does not have a material impact on our results. The risk-aversion parameter is in the bounds of what is usually assumed in the literature ($\gamma = 5$). However, the choice of δ is less obvious because the literature provides little guidance. A notable exception is Ju and Miao (2012), who provide a baseline calibration δ which is 4 times as high as γ. However, we are somewhat more conservative and assume $\delta = 10$. Increasing the ambiguity aversion parameter will only strengthen our results. Finally it's worthwhile to note that we use a negative beta ($\beta_H < 0$). This assumption is in line with our empirical analysis in Section 5. If we switch the sign of beta our findings do not change in absolute terms.

In Figure 1 we show the premium that is generated by the hedge asset for different levels of ξ_{b_H}. In our numerical results we convert cash-flow risk premiums of equations (42) and (82) to a more "natural" percentage risk premium by dividing by equilibrium price. We obtain a risk premium of -6.3% under no beta uncertainty ($\xi_{b_H} = 0$). A one standard deviation change in ξ_{b_H}, ($\xi_{b_H} = 1$) increases the premium in absolute terms by almost 2%. Thus, in our parameterization, the uncertainty premium constitutes to more than 23% of the total premium. To better explain the mechanism at play, we also include Figure 2. In this plot we show what happens to the position of the insurance provider (the uninformed agent) when we increase the level of beta uncertainty. It is sufficient to show the holding of one side of the market, because the other side has the same holding with opposite sign (since the asset is in zero net supply). The main idea is as follows. When beta uncertainty increases, it increases the level of mean uncertainty through (33). This in turn makes the insurance provider more reluctant to hold the asset, as is visible in (27). This decrease in demand (in absolute terms) is visible in Figure 2. This decrease in demand also decreases the price of the hedge asset, and by this - increases the total premium.

Next we consider what happens to the total premium when we introduce the derivative. For simplicity, we assume that the distribution parameters are the same as the long-short portfolio. The only exception is that we make the asset more secure by setting $\xi_{b_D} = 0$[10] and vary ξ_{b_H}. With the addition of a second asset, another parameter of choice is the correlation between the two assets. As was stated before, this is a key parameter in our setting. In the extreme case where the two assets are not correlated, the two markets equilibrate independently, and thus do not affect each other. In Figure 3 we show what happens to the risk premium of the uncertain hedge asset for different levels of beta uncertainty of the long-short portfolio, and correlations between the two hedge assets. Firstly, as was suggested in Theorem (4.3), the change in the premium occurs when there is some degree of parameter uncertainty because $\xi_{b_H} = 0$ results in no change in the risk premium. Secondly, the change in risk premium depends on the level of correlation between

[10]Note that in this case, the level of ξ_{m_D} becomes irrelevant since the uninformed agent learns m_D exactly from prices.

the assets. All else equal, a higher correlation between the long-short portfolio and the derivative implies lower diversification benefits and therefore lower demands. The change in risk premium caused by the introduction of the derivative is not only large in nominal terms, but also in relative terms (approximately 25%)

As with the one asset case, in order to explain the mechanism, we also present the holdings of the insurance provider when she can trade both assets. For this part we fix $r = 0.7$ and vary ξ_{b_H}. Figure 4 shows the result of this exercise. The first observation is that even if there is no hedging uncertainty, the exposure of the insurance provider to the first asset changes when a second one is introduced. This is a result of the diversification benefits that the investor has from investing in two not perfectly correlated assets. Also, when there is no beta uncertainty, the two assets are exactly the same, therefore the positions in the two assets are also equal. However, a similar picture to the one asset case arises when beta uncertainty is increased. For a higher level of beta uncertainty the insurance provider decreases her holding in the long-short portfolio (in absolute terms) and substitutes this with holding in the derivative.

[Figure 1 about here.]

[Figure 2 about here.]

[Figure 3 about here.]

[Figure 4 about here.]

5 Empirical evidence: Inflation premium and TIPS

In this section we test our main theoretical prediction empirically. We hypothesize that - before the availability of TIPS - investors hedge inflation risk in equity markets. More specifically, by buying (shorting) shares with high (low) inflation betas, the investor sets up a portfolio with high payoffs in inflationary regimes and thereby buys protection against inflation risk. If the demand for equity-based inflation hedges is sufficiently high, prices of stocks with high (low) inflation betas are pushed up (down). As a consequence of these price movements, long-short inflation beta portfolios earn a negative premium which is commonly labeled the inflation risk premium.

We hypothesize that this mechanism changes fundamentally once TIPS are introduced to financial markets. Since the main purpose of floating TIPS is trading inflation risk, we believe that the availability of such an asset should affect premiums in equity markets. More specifically, our theory suggests that some investors would rather use TIPS to hedge inflation risk than long-short stock portfolios.[11] In other words, hedging demand moves from equity markets to TIPS. As a natural consequence of this transition, inflation risk premiums in stock markets decrease in absolute terms.

[Table 2 about here.]

[Table 3 about here.]

We test this conjecture in Table 2. More specifically, we sort stocks into portfolios based on their inflation betas. We record returns for a portfolio that takes a long position in the highest inflation beta decile and a short position in the lowest inflation beta decile. We regress returns for the long-short portfolio on a dummy that takes the value of one when TIPS are being traded in secondary markets in January 1997 (Dum_{TIPS}). The positive loading on the TIPS dummy is in line with our theoretical prediction (Theorem 4.3) that the inflation premium decreases in absolute terms. The change in risk premium is economically large (approximately 0.5% per month) and statistically significant. The findings become stronger when we use value-weighted portfolios which is consistent with the notion that investors prefer large and liquid stocks over small ones.

To rule out that our findings are driven by other popular risk factors, we also calculate risk-adjusted returns in Table 3. In this analysis, the sorting remains unchanged, but we now use risk-adjusted returns rather than raw returns. Where risk-adjusted returns are defined as the error terms of a standard Fama and French (1993) three factor regression. The findings corroborate the results in Table 2 and are also in line with our main theoretical predictions. Summarizing, when TIPS are available, inflation risk premiums in stocks markets decline in absolute terms.

[11] These shifts or substitutions in demand are quantified in equations (52), (53), (63) and (64).

This effect should not only be limited to the TIPS market since there are numerous examples where derivatives with direct hedges were introduced. One notable example can be the introduction of options. One problem with using options is that it's not clear which sources of risk they hedge. It can be volatility risk in which agents use straddles to hedge out their exposure. However, some agents can be also exposed to crash risk, and then the proper hedging instrument should be an out of the money put option. Our methodolofy relies on the fact that we can observe the risk factor that the hedge asset hedges so we can avoid the direct measurment of uncertainty. Due to these confounding effects we decided to focus our scope on one application and to leave the analysis of the option market for future research.

6 Conclusion

This paper casts doubt on the conjecture that the stock market is the natural place to hedge risks. Over the past decades many studies have relied on hedging strategies to identify risk factors in the cross-section of stocks (see Harvey, Liu, and Zhu (2016) for an overview). In order for such a risk premium to emerge, there should be sufficiently many investors to move prices and thus create a risk premium. However, from the perspective of an investor, the stock market does not seem to be the natural place to take such positions.

Consider, for example, an investor who wants to trade crash risk. In order to hedge crash risk or reap a crash risk premium in equity markets, the investor would have to estimate betas, create long-short portfolios and re-balance positions to make sure that crash risk exposures attain the desired level. Notwithstanding practical difficulties,[12] investors face substantial uncertainty regarding next period's level of crash risk exposure.[13] In an extreme scenario, next period's crash risk exposure (beta) may even revert sign and thus increase (decrease) risk of a positions that was supposed to hedge risk (reap a crash risk premium). Why would an investor be willing to accept such hedging uncertainty when any desired level of crash risk exposure can easily be obtained in the option market without facing such uncertainty?

[12] Think also about short sale and liquidity constraints.

[13] Barahona, Driessen, and Frehen (2019) indicate that downside betas are highly time-varying and very difficult to predict.

We therefore set up a simple model with two agents who want to hedge exposure to an exogenous shock using risky assets (stocks). On the one hand, the informed agent observes expected payoffs and hedging potential (correlation between payoffs of the risky asset and the exogenous shock). On the other hand, the uninformed agent faces large uncertainty regarding the hedging potential. We further assume smooth ambiguity aversion preferences and determine equilibrium demands, prices and risk premiums. We identify an uncertainty premium that *increases* in hedging (beta) uncertainty.

In a second stage, we expand the investment universe with a derivative (e.g. an option or treasury inflation protected security). The uninformed agent faces substantially less hedging uncertainty for the derivative. We again solve for equilibrium demands, prices and risk premiums and document a re-allocation of wealth from the risky to the derivative. This re-allocation can be contributed to: (i) less hedging uncertainty (ii) diversification benefits. Hence, our theoretical findings are in line with the conjecture that investors prefer markets with less hedging uncertainty when trading (hedging) risks.

In line with these theoretical findings, we show that the inflation risk premium declines by 0.5% per month (in absolute terms) after TIPS markets are opened. This drop is statistically significant and suggests that investors re-allocate hedging demands from the stock market to the TIPS market.

A Appendix

A.1 Proof of proposition 3.2

We first impose an optimal demand function for the informed investor and thus start from equation (18) and also impose that markets equilibrate

$$\left(\frac{m_H - R_F P_H^*}{\gamma\sigma_H^2} - b_H q^{(I)}\right) + \alpha^{(U)} = 0 \tag{85}$$

This allows us to express prices as follows

$$P_H^* = \frac{1}{R_F} m_H - \frac{\gamma\sigma_H^2 q^{(I)}}{R_F} b_H + \alpha^{(U)} \frac{\gamma\sigma_H^2}{R_F}. \tag{86}$$

Substituting (86) into the updating formulas (31), (32), and (33) leads to equilibrium updating rules

$$E\left(m_H|P^*\right) = \eta_H + \frac{\xi_{m_H}^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2}\left[\left(m_H - \eta_H\right) - \gamma\sigma_H^2 q^{(I)}\left(b_H - \beta_H\right)\right] \tag{87}$$

$$Var\left(m_H|P^*\right) = \xi_{m_H}^2 - \frac{\left(\xi_{m_H}^2\right)^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2} \tag{88}$$

$$E\left(b_H|P^*\right) = \beta_H + \frac{\gamma\sigma_H^2 q^{(I)} \xi_{b_H}^2}{\xi_{m_H}^2 + \left(\gamma\sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2}\left[\left(m_H - \eta_H\right) - \gamma\sigma_H^2 q^{(I)}\left(b_H - \beta_H\right)\right] \tag{89}$$

While it may seem counter-intuitive that expectations of m_H and b_H are functions of m_H and b_H, these expressions should be interpreted from an equilibrium perspective. In other words, an uninformed agent's expectations of m_H (b_H) are increasing in the draws of m_H (b_H) because she learns through prices. Finally, we use the optimal demand function for the uninformed type (27) and again the market clearing condition for the hedge asset:

$$0 = \frac{m_H - R_F P_H^*}{\gamma\sigma_H^2} - b_H q^{(I)} + \frac{E\left(m_H|P^*\right) - R_F P_H^*}{\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)} - E\left(b_H|P^*\right) q^{(I)} \frac{\gamma\sigma_H^2}{\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)}. \tag{90}$$

This leads to the following expression for the equilibrium price

$$P_H^* = \frac{1}{R_F}\left\{\frac{\gamma\sigma_H^2\left[m_H + E\left(m_H|P^*\right)\right] + \delta Var\left(m_H|P^*\right)m_H}{2\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)} - \frac{\gamma\sigma_H^2\left[\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)\right]}{2\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)}\left[b_H q^{(I)} + E\left(b_H|P^*\right)q^{(U)}\frac{\gamma\sigma_H^2}{\gamma\sigma_H^2 + \delta Var\left(m_H|P^*\right)}\right]\right\}, \quad (91)$$

which is again a uniquely defined function of the primitives.

For the market asset, the equilibrium is even more straightforward, since the demand function for the market asset is independent of all the other risks for both investors, the equilibrium condition is:

$$\overline{\alpha} = \alpha_M^{(U)} + \alpha_M^{(I)} = \frac{\mu_M - R_F P_M^*}{\gamma\sigma_M^2} + \frac{\mu_M - R_F P_M^*}{\gamma\sigma_M^2} \quad (92)$$

Which is solved by

$$P_M^* = \frac{1}{R_F}\left(\mu_M - \frac{1}{2}\gamma\sigma_M^2\overline{\alpha}\right) \quad (93)$$

A.2 Proof of Proposition 3.3

It is easy to see that if we assume that there is no uncertainty over b. Which means that $b_H = \beta_H$ and $\xi_{b_H}^2 = 0$, the equilibrium updating functions reduce to

$$E\left(m_H|P^*\right) = \eta_H \quad (94)$$

$$E\left(b_H|P^*\right) = \beta_H \quad (95)$$

$$Var\left(m_H|P^*\right) = 0 \quad (96)$$

Substitution into the equilibrium price function leads to

$$P_H^* = \frac{1}{R_F}\left(\eta_H - \frac{\gamma\sigma_H^2}{2}\beta_H\left(q^{(I)} + q^{(U)}\right)\right). \quad (97)$$

A.3 Parameters uninformed agent's optimal demand

We define A_H, A_D, B_H, B_D, C_H and C_D as follows.

$$A_H = \frac{\gamma\sigma_H^2}{\gamma\sigma_H^2 + \delta Var\left(m_H|P\right)} \tag{98}$$

$$A_D = \frac{\gamma\sigma_D^2}{\gamma\sigma_D^2 + \delta Var\left(m_D|P\right)} \tag{99}$$

$$B_H = \frac{1}{\gamma\sigma_H^2 + \delta Var\left(m_H|P\right)} \tag{100}$$

$$B_D = \frac{1}{\gamma\sigma_D^2 + \delta Var\left(m_D|P\right)} \tag{101}$$

$$C_H = \beta_{D,H} A_H + \delta cov\left(m_H, m_D|P\right) B_H \tag{102}$$

$$C_D = \beta_{H,D} A_D + \delta cov\left(m_H, m_D|P\right) B_D. \tag{103}$$

A.4 Parameters equilibrium prices

In this appendix we define L_H, L_D, F_H, F_D, G_H, G_D, K_H and K_D and the conditional expectations of the expected payoffs and betas.

$$L_H = \frac{1}{\gamma\sigma_H^2} \tag{104}$$

$$L_D = \frac{1}{\gamma\sigma_D^2} \tag{105}$$

$$F_H = -\left\{\frac{1}{1 - r_{H,D}^2 A_H A_D}\left[B_H E\left(m_H|P^*\right) - E\left(b_H|P^*\right) q^{(U)} A_H - \beta_{D,H} A_D \left(B_D E\left(m_D|P^*\right) - E\left(b_D|P^*\right) q^{(U)} A_D\right)\right] + \frac{1}{1 - r_{H,D}^2}\left[L_H m_H - b_H q^{(I)} - \beta_{D,H}\left(L_D m_D - b_D q^{(I)}\right)\right]\right\} \tag{106}$$

$$F_D = -\left\{\frac{1}{1 - r_{H,D}^2 A_H A_D}\left[B_D E\left(m_D|P^*\right) - E\left(b_D|P^*\right) q^{(U)} A_D - \beta_{H,D} A_H \left(B_H E\left(m_H|P^*\right)\right) - E\left(b_H|P^*\right) q^{(U)} A_H\right] + \frac{1}{1 - r_{1,2}^2}\left[L_D m_D - b_D q^{(I)} - \beta_{H,D}\left(L_H m_H - b_H q^{(I)}\right)\right]\right\} \tag{107}$$

$$G_H = -R_F \left(\frac{B_H}{1 - r_{H,D}^2 A_H A_D} + \frac{L_H}{1 - r_{H,D}^2} \right) \tag{108}$$

$$G_D = -R_F \left(\frac{B_D}{1 - r_{H,D}^2 A_H A_D} + \frac{L_H}{1 - r_{H,D}^2} \right) \tag{109}$$

$$K_H = \beta_{D,H} R_F \left(\frac{B_D A_H}{1 - r_{1,2}^2 A_H A_D} + \frac{L_D}{1 - r_{H,D}^2} \right) \tag{110}$$

$$K_D = \beta_{H,D} R_F \left(\frac{B_H A_D}{1 - r_{1,2}^2 A_H A_D} + \frac{L_H}{1 - r_{H,D}^2} \right). \tag{111}$$

$$E\left(m_H | P^*\right) = \eta_H + \frac{\xi_{m_H}^2}{\xi_{m_H}^2 + \left(\gamma \sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2} \left[\left(m_H - \eta_H\right) - \gamma \sigma_H^2 q^{(I)} \left(b_H - \beta_H\right) \right] \tag{112}$$

$$E\left(m_D | P^*\right) = \eta_D + \frac{\xi_{m_D}^2}{\xi_{m_D}^2 + \left(\gamma \sigma_D^2 q^{(I)}\right)^2 \xi_{b_D}^2} \left[\left(m_D - \eta_D\right) - \gamma \sigma_D^2 q^{(I)} \left(b_D - \beta_D\right) \right] \tag{113}$$

$$E\left(b_H | P^*\right) = \beta_H + \frac{\gamma \sigma_H^2 q^{(I)} \xi_{b_H}^2}{\xi_{m_H}^2 + \left(\gamma \sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2} \left[\left(\left(m_H - \eta_H\right) - \gamma \sigma_H^2 q^{(I)} \left(b_H - \beta_H\right) \right] \tag{114}$$

$$E\left(b_D | P^*\right) = \beta_D + \frac{\gamma \sigma_D^2 q^{(I)} \xi_{b_D}^2}{\xi_{m_D}^2 + \left(\gamma \sigma_D^2 q^{(I)}\right)^2 \xi_{b_D}^2} \left[\left(m_D - \eta_D\right) - \gamma \sigma_D^2 q^{(I)} \left(b_D - \beta_D\right) \right] \tag{115}$$

$$Var\left(m_H | P^*\right) = \xi_{m_H}^2 - \frac{\left(\xi_{m_H}^2\right)^2}{\xi_{m_H}^2 + \left(\gamma \sigma_H^2 q^{(I)}\right)^2 \xi_{b_H}^2} \tag{116}$$

$$Var\left(m_D | P^*\right) = \xi_{m_D}^2 - \frac{\left(\xi_{m_D}^2\right)^2}{\xi_{m_D}^2 + \left(\gamma \sigma_D^2 q^{(I)}\right)^2 \xi_{b_D}^2}. \tag{117}$$

A.5 Parameters equilibrium risk premium

In this appendix we define $\tilde{F_H}$, $\tilde{F_D}$,$\tilde{G_H}$, $\tilde{G_D}$, $\tilde{K_H}$, $\tilde{K_D}$

$$\tilde{F_H} = -\left[\frac{1}{1-r_{H,D}^2 A_H A_D}\left(-\beta_H q^{(U)} A_H + \beta_{D,H} A_H A_D \beta_D q^{(U)}\right) + \frac{1}{1-r_{H,D}^2}\left(-\beta_H q^{(I)} + \beta_{D,H}\beta_D q^{(I)}\right)\right] \tag{118}$$

$$\tilde{F_D} = -\left[\frac{1}{1-r_{H,D}^2 A_H A_D}\left(-\beta_D q^{(U)} A_D + \beta_{H,D} A_H A_D \beta_H q^{(U)}\right) + \frac{1}{1-r_{H,D}^2}\left(-\beta_D q^{(I)} + \beta_{H,D}\beta_D q^{(I)}\right)\right] \tag{119}$$

$$\tilde{G_H} = \frac{B_H}{1-r_{H,D}^2 A_H A_D} + \frac{C_H}{1-r_{H,D}^2} \tag{120}$$

$$\tilde{G_D} = \frac{B_D}{1-r_{H,D}^2 A_H A_D} + \frac{C_D}{1-r_{H,D}^2} \tag{121}$$

$$\tilde{K_H} = -\beta_{D,H}\left(\frac{B_D A_H}{1-r_{H,D}^2 A_H A_D} + \frac{C_D}{1-r_{H,D}^2}\right) \tag{122}$$

$$\tilde{K_D} = -\beta_{H,D}\left(\frac{B_H A_D}{1-r_{H,D}^2 A_H A_D} + \frac{C_H}{1-r_{H,D}^2}\right) \tag{123}$$

A.6 Proof of Proposition 4.2

Since we showed that $cov\left(m_H, m_D|P\right) = 0$, we can plug this into (80) and (81). The market clearance condition is:

$$\alpha_M^{(U)} + \alpha_M^{(I)} = \overline{\alpha} \tag{124}$$

$$\alpha_H^{(U)} + \alpha_H^{(I)} = 0 \tag{125}$$

$$\alpha_D^{(U)} + \alpha_D^{(I)} = 0 \tag{126}$$

The last two conditions can be expressed as a system of linear equation:

$$G_H P_H^* + K_H P_D^* = F_H \tag{127}$$

$$G_D P_D^* + K_D P_H^* = F_D \tag{128}$$

$$\tag{129}$$

Where $G_H, G_D, K_H, K_D, F_H, F_D$ are defined in A.4. Which is solved by:

$$P_H^* = \frac{F_H G_D - F_D K_H}{G_H G_D - K_H K_D} \tag{130}$$

$$P_D^* = \frac{F_D G_H - F_H K_D}{G_H G_D - K_H K_D} \tag{131}$$

And for the market price:

$$P_M^* = \frac{1}{R_F}\left(\mu_M - \frac{1}{2}\gamma\sigma_M^2\overline{\alpha}\right) \tag{132}$$

A.7 Proof of theorem 4.3

The first part is a consequence of (42). The second part is as follows. Since we assumed that $q^{(I)} - q^{(U)} > 0$. The sign of (84) depends solely on:

$$\beta_{D,H}\beta_D(\delta V_H + 2\gamma\sigma_H^2) - 2r_{H,D}^2\beta_H\gamma\sigma_H^2 \tag{133}$$

Since we assumed $\beta_{D,H} > 0$ and $\beta_H, \beta_D < 0$ then we have a difference of two negative terms. However, the magnitude of the first one depends on δV_H which means that there is some $V_H \in \mathbb{R}$ such that the first term out-weights the second one, and makes the entire expression negative. The last part is easily proven upon setting $V_H = 0$ in (84).

References

Barahona, R., J. Driessen, and R. Frehen, 2019, Can unpredictable risk exposure be priced?, working paper.

Ellsberg, D., 1961, Risk, ambiguity, and the savage axioms, *Quarterly Journal of Economics* 75, 643–669.

Fama, E., and K. French, 1993, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics* 33, 3–56.

Gilboa, I., and D. Schmeidler, 1989, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics* 18, 141–153.

Gollier, C., 2016, Evaluation of long-dated assets: The role of parameter uncertainty, *Journal of Monetary Economics* 84, 66–83.

Harvey, C., Y. Liu, and H. Zhu, 2016, . . . and the cross-section of expected returns, *Review of Financial Studies* 29, 5–68.

Ju, N., and J. Miao, 2012, Ambiguity, learning, and asset returns, *Econometrica* 80, 559–591.

Klibanoff, P., M. Marinacci, and S. Mukerji, 2005, A smooth model of decision making under ambiguity, *Econometrica* 73, 1849–1892.

Lewellen, J., S. Nagel, and J. Shanken, 2008, A skeptical appraisal of asset-pricing tests, *Journal of Financial Economics*, Forthcoming.

Linnainmaa, J., and M. Roberts, 2017, The history of the cross section of stock returns, *Review of Financial Studies* 31, 2606–2649.

Maccheroni, F., M. Marinacci, and D. Ruffino, 2013, alpha as ambiguity: robust mean-variance portfolio analysis, *Econometrica* 81, 1075–1113.

Maenhout, P., 2004, Robust portfolio rules and asset pricing, *Review of Financial Studies* 17, 951–983.

McLean, R., and J. Pontiff, 2016, Does academic research destroy stock return predictability?, *Journal of Finance* 71, 5–32.

Ruffino, D., 2014, A robust capital asset pricing model, working paper.

Vasicek, O. A., 1973, A note on using cross-sectional information in bayesian estimation of security betas, *Journal of Finance* 28, 1233–1239.

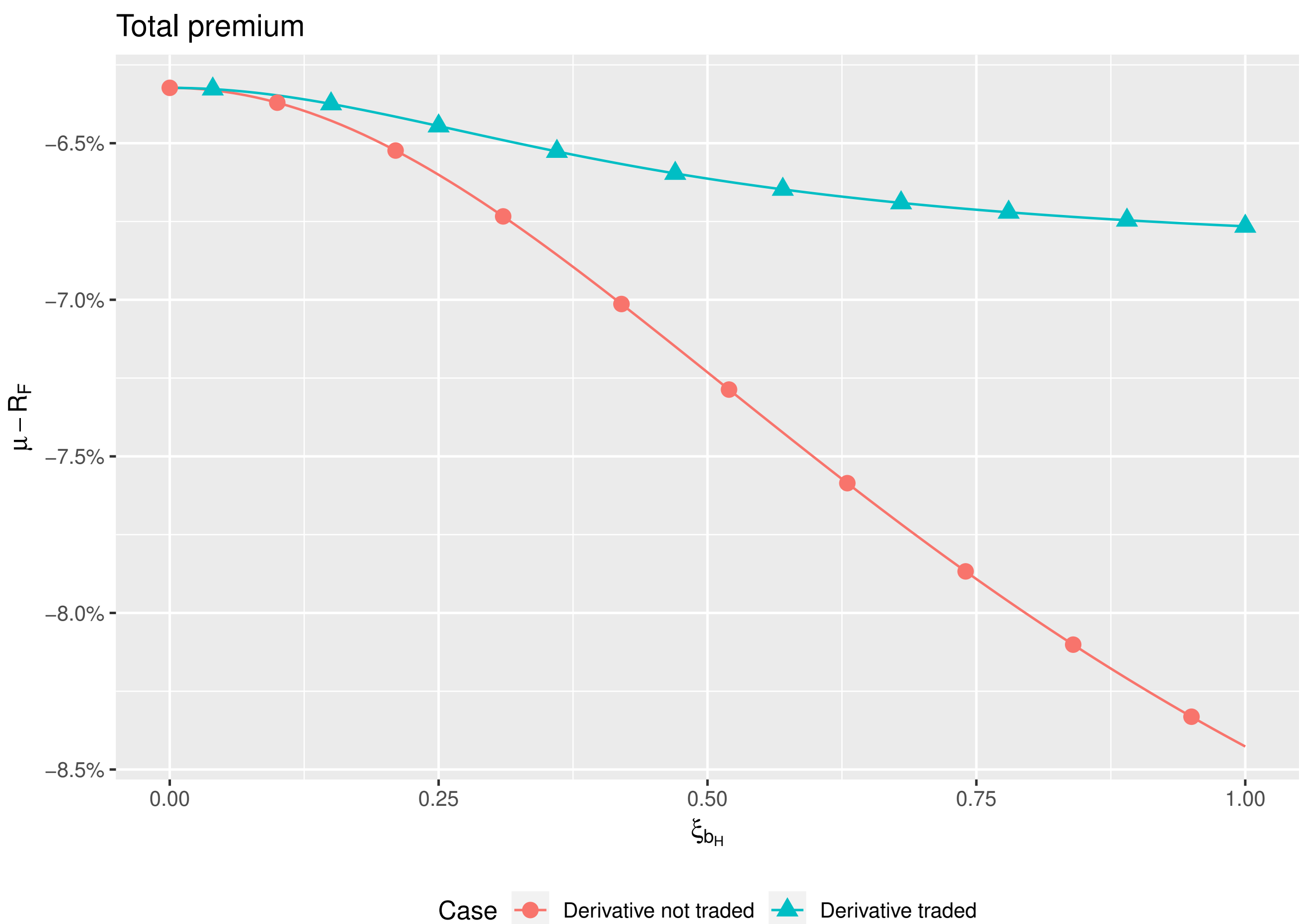

Figure 1: This figure shows the risk premium as a function of beta uncertainty in the case where the derivative is traded (triangle) and when it is not traded (dot). We assume a correlation between the assets of 0.9. On the x-axis is beta uncertainty of the long-short portfolio ξ_{b_H} and on the y-axis is the percentage premium paid by the uncertain hedge asset $\frac{\eta_H - R_F P_H^*}{P_H^*}$. The premium is negative because we consider $\beta_H < 0$. The difference between the two lines is the drop (in absolute terms) of the premium due to the inclusion of the derivative.

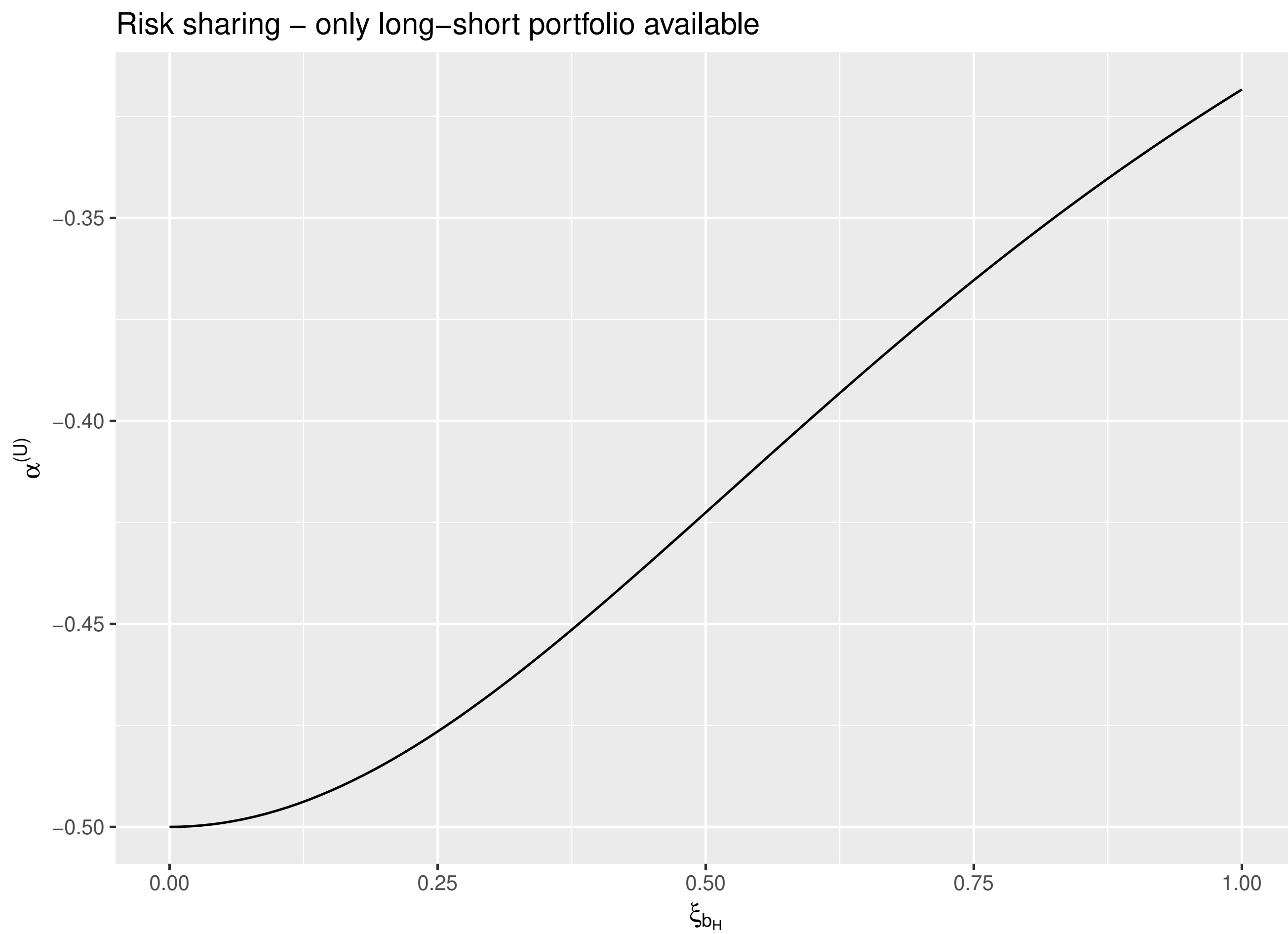

Figure 2: This figure shows the equilibrium position of the insurance provider (the uninformed agent) in the long-short portfolio when there is no derivative in the market. On the x-axis is the uncertainty over the beta ξ_{b_H} and on the y-axis is the equilibrium position $\alpha_H^{(U)^*}$. Since we consider the case of $\beta < 0$, the insurance provider sells the asset and reaps the (negative) premium.

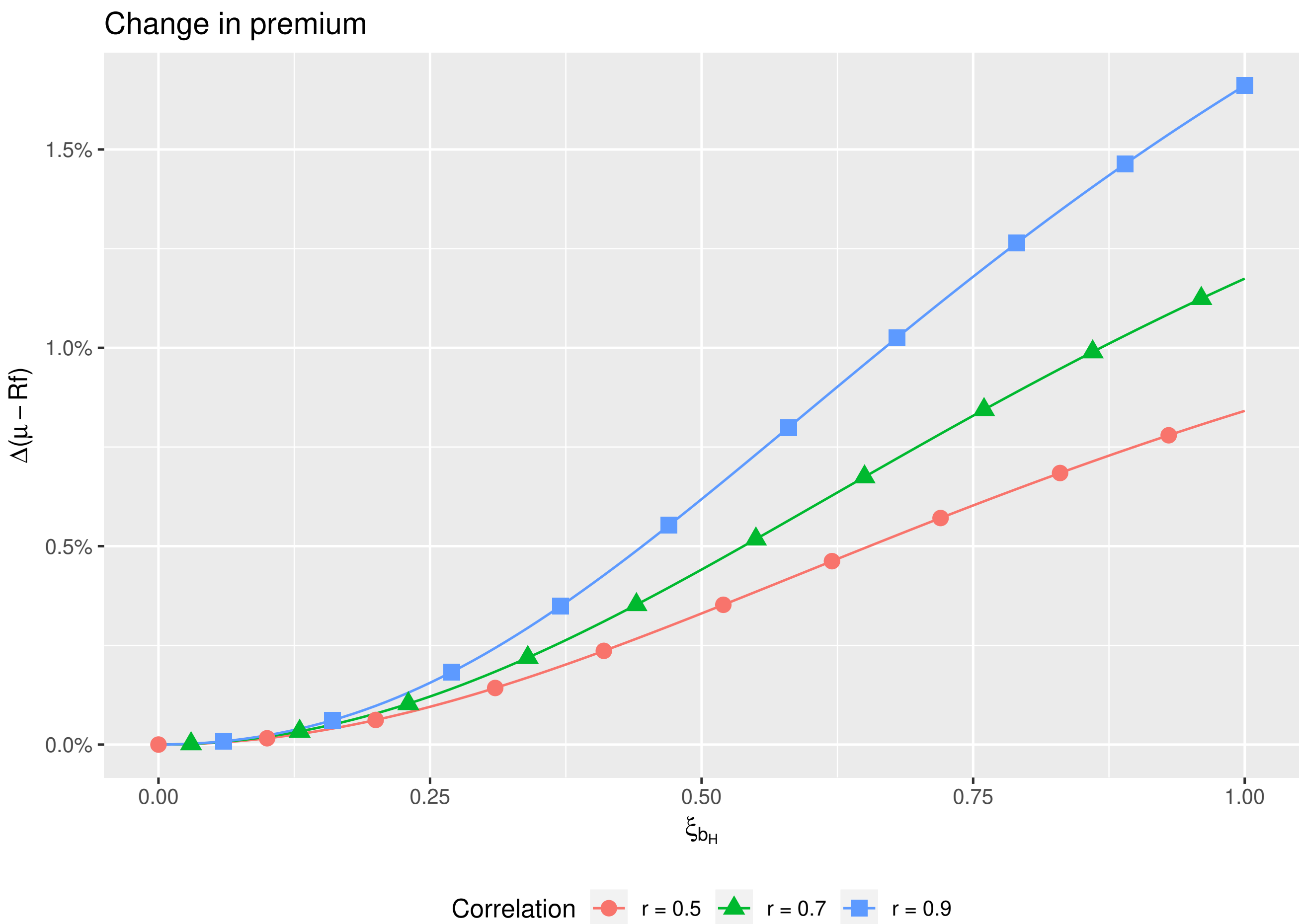

Figure 3: This figure shows the change in the total premium of the uncertain hedge asset when a derivative is introduced into the market. On the x-axis is the uncertainty over the hedge potential ξ_{b_H} and on the y-axis is the change in the percentage premium paid by the hedge asset $\Delta\frac{\eta_H - R_F P_H^*}{P_H^*}$ Different lines constitute to different levels of correlation between the two assets.

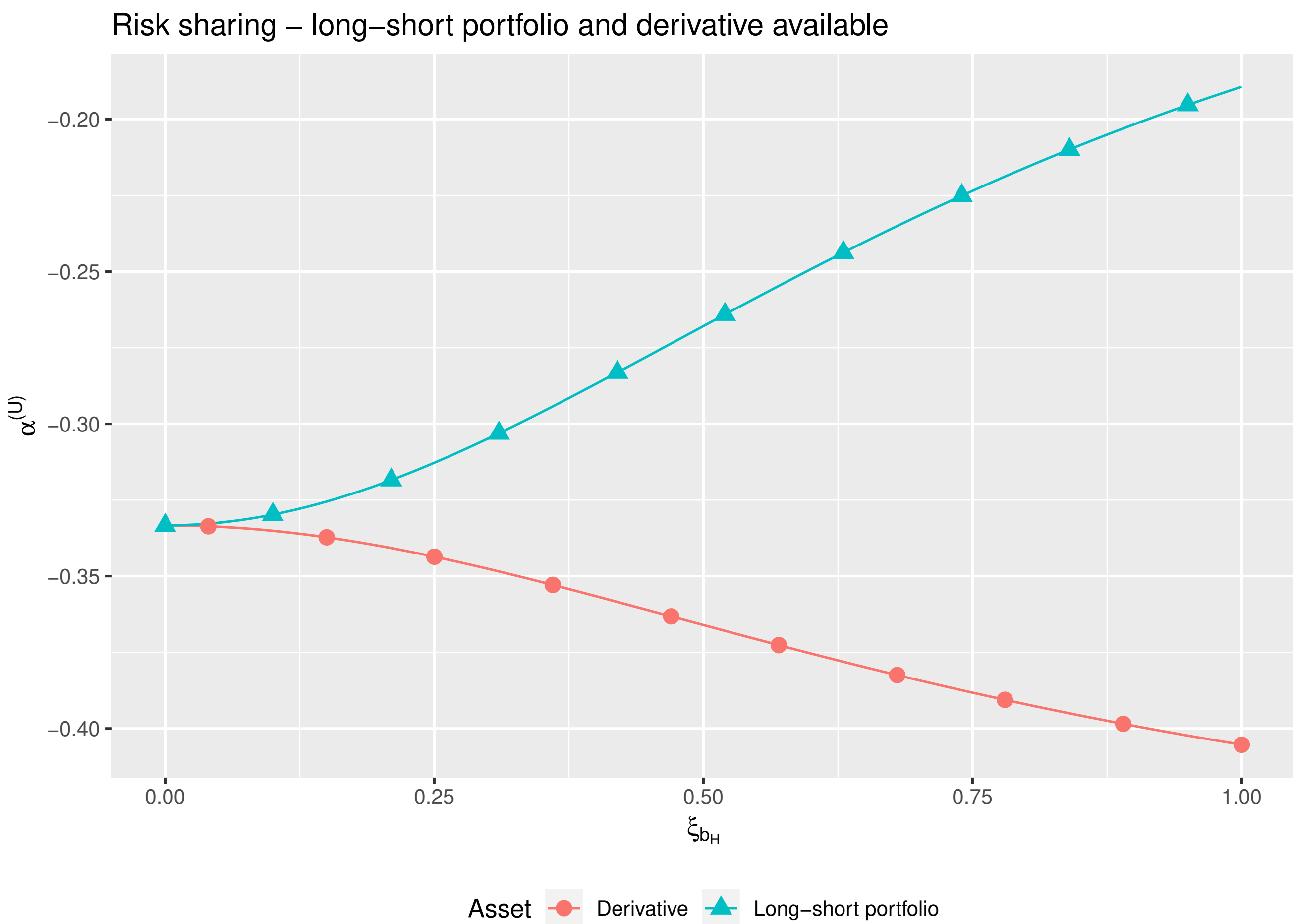

Figure 4: This figure shows the equilibrium position of the insurance provider (the uninformed agent) in both the derivative and long-short portfolio when both are available. On the x-axis is the uncertainty over the hedge potential ξ_{b_H} and on the y-axis is the equilibrium position $\alpha_H^{(U)*}$ and $\alpha_D^{(U)*}$

Table 1: **Parameters**

In this table we present the parameter values we used in the numerical analysis.

Parameter	Symbol	Value
Gross risk-free rate	R_F	1.00
Risk-aversion	γ	5.00
Ambiguity aversion	δ	10.00
Prior expectation of the mean	η_H	1.2
Prior s.d of the mean	ξ_{m_H}	0.25
Volatility	σ_H	0.18
Prior beta expectation	β_H	-1
Prior s.d of beta	ξ_{b_H}	1
Informed agent's exposure to risk	$q^{(I)}$	1
Uninformed agent's exposure to risk	$q^{(U)}$	0

Table 2: **Inflation risk premium and TIPS - raw returns**

In this table we regress monthly excess returns of a long-short inflation beta portfolio on a dummy for the introduction of TIPS (Dum_{TIPS}) that takes the value of one after December 1996. Stocks are sorted into decile portfolios on their beta with respect to changes in the CPI. We take the difference between excess returns of stocks that are in the top beta decile and the bottom decile. In columns 1 and 2 we sort on raw betas (β_{CPI} - raw) that are estimated using a linear regression of a moving window of 60 months. In columns 3 and 4 we sort on betas that are shrunk using Vasicek (1973)'s shrinkage estimator on the entire sample (β_{CPI} - Vasicek-full). In columns 5 and 6 we sort on Vasicek betas estimated over an expanding window using information available at time t only (β_{CPI} - Vasicek-window). In columns 1, 3 and 5 we compute equally-weighted portfolio returns, and in columns 2, 3 and 6 portfolio returns are value-weighted using market capitalization. The standard errors are corrected for heteroskedasticity and are presented in the parentheses.

	β_{CPI} - raw		β_{CPI} - Vasicek-full		β_{CPI} - Vasicek-window	
	EW	VW	EW	VW	EW	VW
Dum_{TIPS}	0.355	1.319***	0.592**	1.150***	0.423*	0.944**
	(0.253)	(0.389)	(0.254)	(0.379)	(0.244)	(0.367)
intercept	−0.327*	−0.810***	−0.470***	−0.718***	−0.351**	−0.775***
	(0.170)	(0.243)	(0.172)	(0.225)	(0.169)	(0.230)
Observations	677	677	677	677	677	677
R^2	0.003	0.017	0.008	0.014	0.004	0.010
Adjusted R^2	0.001	0.015	0.006	0.012	0.003	0.008

*p<0.1; **p<0.05; ***p<0.01

Table 3: **Inflation risk premium and TIPS - risk-adjusted returns**

In this table we regress monthly risk-adjusted returns of a long-short inflation beta portfolio on a dummy for the introduction of TIPS (Dum_{TIPS}) that takes the value of one after December 1996. Risk-adjusted returns are defined as a residual of a firm-specific regression of excess stock returns on the excess market return, SMB and HML (obtained from Kenneth French's website). Stocks are sorted into decile portfolios on their beta with respect to changes in the CPI. We take the difference between risk-adjusted returns of stocks that are in the top beta decile and the bottom decile. In columns 1 and 2 we sort on raw betas (β_{CPI} - raw) that are estimated using a linear regression of a moving window of 60 months. In columns 3 and 4 we sort on betas that are shrunk using Vasicek (1973)'s shrinkage estimator on the entire sample (β_{CPI} - Vasicek-full). In columns 5 and 6 we sort on Vasicek betas estimated over an expanding window using information available at time t only (β_{CPI} - Vasicek-window). In columns 1, 3 and 5 we compute equally-weighted portfolio returns, and in columns 2, 3 and 6 portfolio returns are value-weighted using market capitalization. The standard errors are corrected for heteroskedasticity and are presented in the parentheses.

	β_{CPI} - raw		β_{CPI} - Vasicek-full		β_{CPI} - Vasicek-window	
	EW	VW	EW	VW	EW	VW
Dum_{TIPS}	0.362	1.341***	0.593**	1.138***	0.434*	0.976***
	(0.256)	(0.396)	(0.257)	(0.384)	(0.246)	(0.371)
intercept	−0.067	−0.549**	−0.242	−0.462**	−0.115	−0.451**
	(0.155)	(0.221)	(0.148)	(0.199)	(0.150)	(0.200)
Observations	677	677	677	677	677	677
R^2	0.003	0.018	0.008	0.014	0.005	0.011
Adjusted R^2	0.002	0.017	0.007	0.013	0.003	0.010

*p<0.1; **p<0.05; ***p<0.01